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The Convex Feasibility Problem and Mann-type Iteration

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The algorithms for solving convex feasibility problems receive great attention for their extraordinary utility and broad applicability in many areas of classical mathematics and modern physical sciences, (for example: computerized tomography). Usually, the convex feasibility problems are solved by projection algorithms. The projection method is a particular case of the Mann iteration process. Numerical examples are given.

Keywords*: convex feasibility problem, projection method, relaxation algorithm, Mann iteration.*

1. Introduction

In the following H is a real Hilbert space (scalar product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$), C is a closed convex subset of H, T : C → C is a nonlinear mapping. We suppose everywhere that the set of fixed points of T is nonempty, Fix(T) $\neq \emptyset$.

The Mann iteration, or Krasnoselki-Mann iteration, has the form

$$x_{n+1} = (1 - t_n)x_n + t_n T(x_n),$$

where T : C → C, C is a closed convex subset of the real Hilbert space H and $\{t_n\} \in$ R is the control sequence.

An application of the Mann-type iteration is the projection algorithms for solving the convex feasibility problem, particularly because such algorithm has the form $x_{n+1} = (1 - t_n)x_n + t_n T(x_n),$ and the projection mapping has the properties required in the Mann-type iteration.

The geometric idea of the projection method is to project the current iteration onto certain set from the intersecting family and to take the next iteration on the straight line connecting the current iteration and this projection. A weight factor gives the exact position of the next iteration. Different strategies concerning the selection of the set onto which the current iteration will be projected, will give particular projection types algorithms.

The objective of this paper is to analyze the projection algorithm as a particular case of Mann iteration. Section 2 deals with convex feasibility problem and the relaxation algorithm. In Section 3 are presented some examples. Finally there are presented the conclusions.

2. The convex feasibility problem

The general convex feasibility problem can be formulated in a very simple way: Given C_i , i=1,..., m a family of closed convex sets with nonempty intersection, $\cap C_i \neq \emptyset$, in a real Hilbert space. Find a point $x \in \cap C_i$.

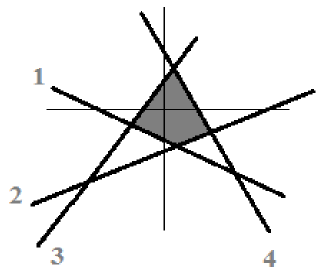

Figure 1. The convex feasibility problem

Let x be a point in H and let $P(x,i)$ be the projection of x onto C_i (if $x \in C_i$ then $P(x,i) = x$).

Let i_x be the least index such that $\|x - P(x,i_x)\| = \max i \|x - P(x,i)\|$ define the mapping $T: H \to H$ by $T(x) = P(x,i_x)$. It is clear that $x \in \cap C_i$ if and only if T (x) = x, hence if and only if x is a fixed point of T , that is $\cap C_i$ = F ix(T).

In a particular case, the sets of family are given by a finite number of half-spaces defined by linear inequalities $\langle a_i, x\rangle + b_i \geq 0, i = 1, 2, \dots, m$. A point which satisfies all these inequalities is a "feasible point" and it can be used for starting the iteration process.

A first result for solving the convex feasibility problem in this case was given by Agmon, Motzkin and Schoenberg in 1954, [1], [10].

The interest for the convex feasibility problem is motivated by some important applications in specific areas. In the paper [2] are pointed out the main applications both in other mathematical algorithms and directly in some practical problems.

For example:

Best Approximation Theory with applications in linear prediction theory (statistics), partial differential equations (Dirichlet problem), complex analysis (Bergman kernels, conformal mappings);

Subgradient algorithms with application in solution of convex inequalities, minimization of convex nonsmooth functions;

Discrete Image Reconstruction with applications in electron microscopy, computerized tomography;

Continuous Image Reconstruction with applications in minimization of convex nonsmooth function, convex inequalities.

Usually, the convex feasibility problems are solved by projection algorithms.

2.1. The projection algorithms

Projection Algorithm is, in it essence, very simple: starting with an initial point (usually arbitrary in the space), the current iteration x_k is projected on certain set of the family, and then the next iteration point is taken on the line segment (x_k, x_k^S), where x_k^S **is** the symmetric of x_k with respect to its projection.

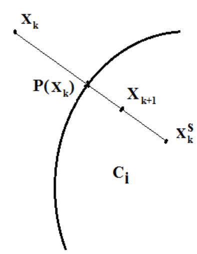

Figure 2. The projection algorithm

There exist two basic strategies to choose the set C_{i_n} on which the current iteration x_k is projected:

(1) the cyclic strategy (the projection is performed successively on all sets of the family in some order): it is obtained the Cyclic Projection Algorithm.

(2) the maximum strategy (at every step the farthest set to the current iteration is chosen); we obtain the Maximum Projection Algorithm.

A complete and exhaustive study of algorithms for solving convex feasibility problem, including comments about their applications and an excellent bibliography, was given by H.H. Bausche and J.M. Borwein [2]. A general form of the projection algorithm is presented in [2] in which the strategy is given by a numerical matrix $\Lambda = \left(\lambda_i^n\right)_{i=1}^m \quad \lambda_i \in [0,1]$, $\sum_{i=1}^m = 1$

and a detailed analysis including strong and weak convergence properties was done for this general algorithm and for its various particular cases.

A typical result is like that of Corollary 4.17, [2]: Suppose C_{i_n} is determined with a suitable strategy (that is the matrix satisfies some properties) and that H is finite dimensional or the interior of $\cap C_i$ is nonempty. Then the sequence $\{x_n\}$ given by this general projection algorithm converges in norm to some point $x \in \cap C_i$.

The projection algorithm was used in [1], [10] for solving a system of linear inequalities (the authors referred to their method as *relaxation algorithm*).

Generalizations for convex sets in real *n*-dimensional spaces were given in [5], [8].

The projection method is a particular case of the Mann iteration process.

The convergence properties of the projection algorithm for convex feasibility problem are obtained from the general convergence properties of the Mann iteration.

2.2. The relaxation algorithm

The Relaxation Method is, in fact, the projection algorithm applied to a particular family of closed convex sets consisting of semi-spaces defined by linear inequalities. This method was investigated in two paper published in 1954 [1, 10] and in our best knowledge, they are among the first results on the projection algorithm for solving a convex feasibility problem. The convergence of this method can be obtained as an application of Theorem 3.4 [9,4].

Consider the following system of linear inequalities:

$\Delta_i(x) = \langle a_i, x \rangle + b_i \geq 0, \quad i=1,2,...,I$

where $a_i, b_i \in E^m$. (Euclidean space).

Every inequality defines a semi-space S_i and we assume that $\cap S_i \neq \emptyset$. If x is any point in E^m we chose a semi-space S_{i_x} which satisfies the conditions:

- S_{i_x} does not contain x
- S_{i_x} is the farthest semi-space to x, it satisfies:

$$|\Delta_{i_x}(x)| = \max_{\Delta_i(x)<0} |\Delta_i(x)|.$$

and, if there exist several semi-spaces satisfying this two condition, we select one who had the smallest index.

Let $P_{i_x}(x)$ be the projection of x onto the hyperplane $\Delta_{i_x}(x) = 0$. If x_n is the current iteration in Relaxation Method, then the next iteration is given by $x_{n+1} = (1-t_n)x_n + t_n P_{i_{x_n}}$, the Mann iteration with T the above projection mapping $x \to P_{i_x}(x)$.

3. Examples

Example 3.1.(Linear inequalities)

Let $f:\mathbf{R}^2 \to \mathbf{R}$, defined by
$f_1(x) = -x + y + 2, f_2(x) = 3x + y + 2, f_3(x) = -0.5x - y + 1, f_1 \geq 0, \quad f_2 \geq 0, f_3 \geq 0$

The initial point is $x = \begin{pmatrix} -4 \\ -5 \end{pmatrix}$ and t=2.

The solution obtained after we applied the relaxation method is:

$$p = \begin{pmatrix} -4 & -5 \\ 5 & -2 \\ 0 & 3 \\ -1.6 & -0.2 \\ 0.2 & 0.4 \end{pmatrix}$$

The geometric solution is to draw a normal line from the current iteration onto certain set form the intersection family and to take the next iteration on this line

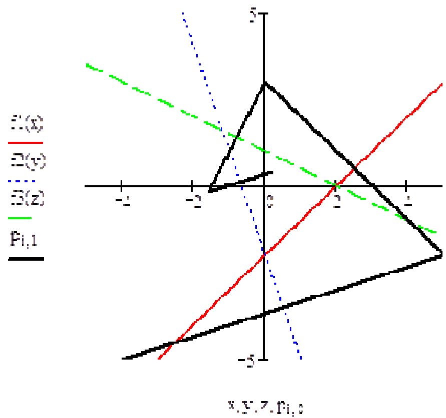

Figure 3.

Example 3.2. For the general convex feasibility problem

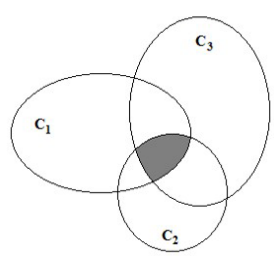

Figure 4.

We consider the following closed convex subsets:

$ec_1(a,x,y) = 2x^2 - xy + 5y^2 - 5x + 3y + 0.5,$
$ec_2(a,x,y) = 6x^2 - xy + 2y^2 + 13x + 3y + 1$
$ec_3(a,x,y) = 5x^2 - 2xy + 2y^2 + x - 2y + 2,$
$ec_1(a,x,y) \geq 0, ec_2(a,x,y) \geq 0, ec_3(a,x,y) \geq 0$

The initial point is $x_0 = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$ and t=1.8.

The solution obtained after the projection algorithm is applied is:

$$p = \begin{pmatrix} -2 & 2 \\ 1.658 & -1.29 \\ -1.068 & -0.851 \\ -0.224 & -0.039 \\ -0.064 & -0.104 \end{pmatrix}$$

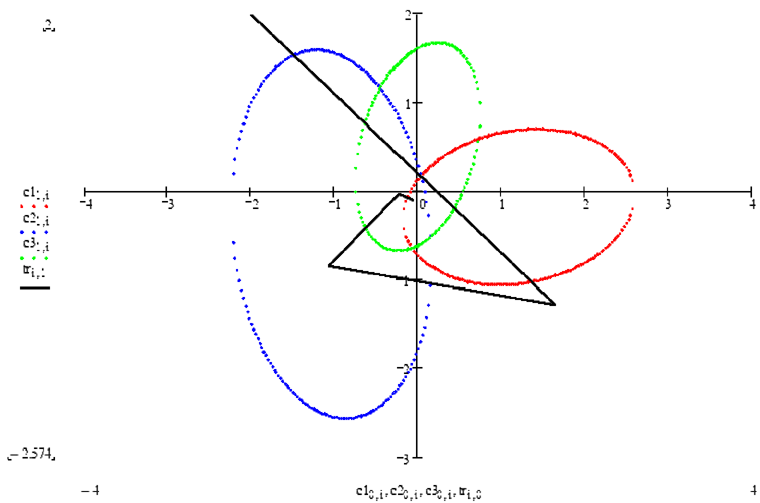

Figure 5.

Example 3.3. The projection algorithm with deviation

We consider the following closed convex subsets:

$ec(a, x, y) = -a_0x^2 - a_1xy - a_2y^2 - a_3x - a_4y + a_5,$
$ec(a, x, y) \geq 0$

with nonempty intersection.

Find a point of the intersection of this family.

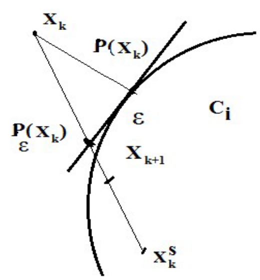

Figure 6.

Starting with an initial point x_0, the current iteration x_k is projected on the set of the family. We move the projection of the point P (x, i), i = 1, ..., n along the tangent to the convex subsets with a given ϵ and then the next iteration is taken on the line segment (x_k, x_k^S), where x_k^S is the symmetric of x_k with respect to the new projection $P(x + \epsilon, i)$.

For the coefficient matrix A:

$$A = \begin{bmatrix} 2 & -1 & 5 & -5 & 3 & 0.5 \\ 6 & 5 & 6 & 9 & 2 & 1 \\ 5 & -2 & 2 & 1 & -2 & 2 \end{bmatrix}$$

and the control sequence t = 1.8 we applied this algorithm to different initial points. The conclusion is that it is obtained a good solution for $\epsilon = 0.01, \ldots 0.20$.

Table 1. Result 1 of Example 3.3

x_0	ϵ	It.	last value
(−2, 2)	0.01	6	(-0.125,-0.068)
(−2, 2)	0.10	9	(0.083,-0.333)
(−2, 2)	0.20	56	(0.083,0.095)
(−2, 2)	0.22	55	(-0.281,1.894)

Table 2. Result 2 of Example 3.3.

x_0	ϵ	It.	last value
(1, 2)	0.01	5	(-0.04,-0.525)
(1, 2)	0.10	16	(-0.034,-0.098)
(1, 2)	0.21	57	(0.171, 0.058)
(1, 2)	0.22	54	(0.233, 1.346)

Table 3. Result 3 of Example 3.3.

x_0	ϵ	It.	last value
(2, −9)	0.01	6	(0.085,-0.154)
(2, −9)	0.10	6	(-0.082,-0.505)
(2, −9)	0.20	37	(0.097,-0.489)
(2, −9)	0.21	54	(0.654,-8.323)

Table 4. Result 4 of Example 3.3.

x_0	ϵ	It.	last value
(−2.5, −10)	0.01	8	(-0.065, -0.136)
(−2.5, −10)	0.10	10	(0.089,-0.327)
(−2.5, −10)	0.20	28	(0.124,-0.296)
(−2.5, −10)	0.21	53	(-0.028,2.943)

4. Conclusions

This paper is a topical one this can be seen from the large number of papers published on this topic.

As the projection algorithm is a particular case of the Mann iteration, the problem is to investigate the conditions under which the Mann iteration for a mapping which is demicontractive and demiclosed at zero converges to a fixed point. This will be studied in a future work.

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