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Equilibrium without Friction of a Particle on a Mobile Surface with Bilateral Constraints

In this paper we will study the equilibrium of a particle on a mobile surface in the case characterized by bilateral constraints between the particle and the surface, and the absence of friction. Based on our previous work, the conditions for the equilibrium are obtained. We prove that the positions of equilibrium on a mobile surface are no longer the same with those obtained for a fixed surface, the system could have either other equilibrium positions, completely different, or some more equilibrium positions, or no equilibrium position.

Keywords: *mobile surface, equilibrium position, conditions of equilibrium*

1. Introduction

The dynamics without friction of the particle on a mobile surface with bilateral constraints was discussed in our previous work [4]. The approach was a multibody type one similar to that presented in [2]. An equivalent approach [3] was also presented. The case of the fixed surface [1] is a particular case.

Let us consider that the particle of mass m and acted by the resultant force

$$\mathbf{F} = F_X\mathbf{i} + F_Y\mathbf{j} + F_Z\mathbf{k}\,, \tag{1}$$

is situated on the mobile surface of equation

$$f\left(X, Y, Z, t\right) = 0\,, \tag{2}$$

where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the unit vectors of the axes OX, OY, and OZ, respectively, F_X, F_Y, and F_Z are the components of the resultant force on the axes OX, OY, and OZ, respectively, while t is the time.

In our previous work [4] we obtained the matrix equation of motion

$$\begin{bmatrix} [\mathbf{m}] & -[\mathbf{B}]^{\mathrm{T}} \\ [\mathbf{B}] & 0 \end{bmatrix} \begin{bmatrix} \{\ddot{\mathbf{q}}\} \\ \lambda \end{bmatrix} = \begin{bmatrix} \{\mathbf{F}\} \\ \{\dot{\mathbf{C}}\} - [\dot{\mathbf{B}}]\{\dot{\mathbf{q}}\} \end{bmatrix}, \tag{3}$$

where

$$[\mathbf{m}] = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}, \tag{4}$$

$$[\mathbf{B}] = \begin{bmatrix} \frac{\partial f}{\partial X} & \frac{\partial f}{\partial Y} & \frac{\partial f}{\partial Z} \end{bmatrix}, \tag{5}$$

$$\{\mathbf{C}\} = \left\{ -\frac{\partial f}{\partial t} \right\}, \tag{6}$$

$$\{\mathbf{F}\} = [F_X \;\; F_Y \;\; F_Z]^{\mathrm{T}}, \tag{7}$$

$$\{\mathbf{q}\} = [X \;\; Y \;\; Z]^{\mathrm{T}}. \tag{8}$$

2. Equilibrium conditions

The parameter λ (Lagrange multiplier) is given by [4]

$$\lambda = \left[[\mathbf{B}][\mathbf{m}]^{-1}[\mathbf{B}]^{\mathrm{T}}\right]\left\{\{\dot{\mathbf{C}}\} - [\dot{\mathbf{B}}]\{\dot{\mathbf{q}}\} - [\mathbf{B}][\mathbf{m}]^{-1}\{\mathbf{F}\}\right\} \tag{9}$$

Let us assume that the point P of coordinates X_P, Y_P and Z_P situated on the mobile surface of equation (2) is an equilibrium point. It results that the coordinates X_P, Y_P, Z_P, and their derivatives are always known; hence, λ is always known (according to the formula (9)).

Moreover, from the relation (3) we get

$$[\mathbf{m}]\{\ddot{\mathbf{q}}\} - [\mathbf{B}]^{\mathrm{T}}\lambda = \{\mathbf{F}\}, \tag{10}$$

and, taking into account that $\{\ddot{\mathbf{q}}\}$ is also known, it results that the equation (10) leads to the expression of the force $\mathbf{F}$.

The inverse problem, that is, one knows the expression of the force $\mathbf{F}$ and asks for the equilibrium positions, is more difficult (in fact, it is a difficult problem even in the case of the fixed surface) and must be discussed depending on each particular case.

A few examples will clarify the problem.

3. Example 1

The particle is situated on the mobile sphere of equation

$$f(X, Y, Z, t) = (X - a\cos\omega t)^2 + Y^2 + Z^2 - R^2 = 0 \tag{11}$$

and is acted by a force $\mathbf{F}$ given by

$$\mathbf{F} = F_X \mathbf{i}. \tag{12}$$

Determine the equilibrium positions and the normal reaction.

Denoting

$$b_1 = \frac{\partial f}{\partial X}, \; b_2 = \frac{\partial f}{\partial Y}, \; b_3 = \frac{\partial f}{\partial Z}, \tag{13}$$

we obtain the equations of motion

$$m\ddot{X} - b_1\lambda = F_X, \; m\ddot{Y} - b_2\lambda = 0, \; m\ddot{Z} - b_3\lambda = 0. \tag{14}$$

Taking into account the expression (11), we obtain that $Y = \mathrm{ct}$, $Z = \mathrm{ct}$ (at the equilibrium the coordinates Y and Z are constant); hence $\ddot{Y} = 0$, $\ddot{Z} = 0$, and the second and the third equations (14) become

$$b_2\lambda = 0, \; b_3\lambda = 0. \tag{15}$$

There are two possible cases.

The first one implies

$$b_2 = b_3 = 0, \tag{16}$$

λ being arbitrary, that is (recall the expressions (13))

$$2Y = 2Z = 0, \tag{17}$$

wherefrom

$$Y = Z = 0. \tag{18}$$

From the equation (11) we have

$$(X - a\cos\omega t)^2 = R^2, \tag{19}$$

wherefrom

$$X = \pm R + a\cos\omega t, \tag{20}$$

$$\dot{X} = -a\omega\sin\omega t, \tag{21}$$

$$\ddot{X} = -a\omega^2\cos\omega t. \tag{22}$$

Replacing now these expressions in the first relation (14), we get

$$-ma\omega^2\cos\omega t - 2(\pm R + a\cos\omega t)\lambda = F_X, \tag{23}$$

$$\lambda - \frac{-F_X - ma\omega^2\cos\omega t}{2(\pm R + a\cos\omega t)}. \tag{24}$$

One may easily observe that the parameter λ and, consequently, the normal reaction N,

$$N = \lambda\frac{\partial f}{\partial X} = -F_X - ma\omega^2\cos\omega t, \tag{25}$$

have not constant values as in the case of the fixed surface.

The second case is characterized by $\lambda = 0$. In this situation, the first equation (14) leads to

$$m\ddot{X} = F_X. \tag{26}$$

Again, $Y = \mathrm{ct}$, $Z = \mathrm{ct}$, and the equation (11) offers

$$X = K + a\cos\omega t, \tag{27}$$

where K is a constant,

$$K = \pm\sqrt{R^2 - Y^2 - Z^2}\,. \tag{28}$$

It results

$$\dot{X} = -a\omega \sin \omega t\,, \tag{29}$$

$$\ddot{X} = -a\omega^2 \cos \omega t \tag{30}$$

and the equilibrium is possible if and only if the force F_X is given by

$$F_X = -ma\omega^2 \cos \omega t\,. \tag{31}$$

4. Example 2

The particle is situated on the same mobile sphere described by the equation (11), but it is acted by its own weight,

$$\mathbf{F} = -mg\mathbf{k}\,. \tag{32}$$

One asks for the equilibrium positions.

We obtain the equations of motion

$$m\ddot{X} - b_1\lambda = 0\,,\ m\ddot{Y} - b_2\lambda = 0\,,\ m\ddot{Z} - b_3\lambda = -mg\,. \tag{33}$$

Again, $Y = \text{ct}$, $Z = \text{ct}$; hence $\ddot{Y} = 0$, $\ddot{Z} = 0$.

The relation $\lambda = 0$ is not a convenient one due to the third relation (33) which would lead to $g = 0$.

From the third relation (33) we get

$$-2\lambda Z = -mg\,, \tag{34}$$

$$\lambda = \frac{mg}{2Z} = \text{ct}\,. \tag{35}$$

The second relation (33) offers

$$2\lambda Y = 0\,, \tag{36}$$

wherefrom

$$Y = 0\,. \tag{37}$$

The equation of the sphere reads now

$$(X - a\cos\omega t)^2 = R^2 - Z^2 = K^2\,, \tag{38}$$

where K is a constant.

It results

$$X = \pm K + a\cos\omega t\,, \tag{39}$$

$$\dot{X} = -a\omega \sin \omega t\,, \tag{40}$$

$$\ddot{X} = -a\omega^2 \cos \omega t \tag{41}$$

and the first equation (33) reads

$$a\omega^2 \cos\omega t + \frac{g}{Z}(\pm K + a\cos\omega t) = 0\,, \tag{42}$$

wherefrom

$$Z = \frac{-g(\pm K + a\cos\omega t)}{a\omega^2 \cos\omega t}. \tag{43}$$

Since $Z = \text{ct}$, we obtain $a = 0$, which is absurd.

In conclusion, there is no equilibrium position in this case.

5. Example 3

Let us assume that the particle is situated on the surface

$$f(X, Y, Z, t) = X \sin(\theta_0 \cos t) + Y \cos(\theta_0 \cos t) = 0, \tag{44}$$

that is, a vertical plan that contains the axis OZ and makes with the axis OX an angle

$$\theta = \theta_0 \cos t. \tag{45}$$

The particle is acted by a force $\mathbf{F}$.

One asks for the conditions in which the equilibrium position is given by the point P of coordinates

$$X_P = d\cos(\theta_0 \cos t),\ Y_P = d\sin(\theta_0 \cos t),\ Z_P = 0. \tag{46}$$

We have to work with the formula (10).

One successively obtains

$$\dot{X}_P = d\theta_0 \sin(\theta_0 \cos t)\sin t, \tag{47}$$

$$\ddot{X}_P = -d\theta_0^2 \sin^2 t \cos(\theta_0 \cos t) + d\theta_0 \cos t \sin(\theta_0 \cos t), \tag{48}$$

$$\dot{Y}_P = -d\theta_0 \sin t \cos(\theta_0 \cos t), \tag{49}$$

$$\ddot{Y}_P = -d\theta_0^2 \sin^2 t \sin(\theta_0 \cos t) - d\theta_0 \cos t \cos(\theta_0 \cos t), \tag{50}$$

$$\dot{Z}_P = 0, \tag{51}$$

$$\ddot{Z}_P = 0, \tag{52}$$

$$\frac{\partial f}{\partial X} = \sin(\theta_0 \cos t), \tag{53}$$

$$\frac{\partial f}{\partial Y} = \cos(\theta_0 \cos t), \tag{54}$$

$$\frac{\partial f}{\partial Z} = 0, \tag{55}$$

$$\frac{\partial f}{\partial t} = -X_P \cos(\theta_0 \cos t)\sin t + Y_P \sin(\theta_0 \cos t)\sin t, \tag{56}$$

$$[\mathbf{B}] = [\sin(\theta_0 \cos t)\ \cos(\theta_0 \cos t)\ 0], \tag{57}$$

$$[\dot{\mathbf{B}}] = [-\cos(\theta_0 \cos t)\sin t\ \ \sin(\theta_0 \cos t)\sin t\ \ 0], \tag{58}$$

$$\{\mathbf{C}\} = \left\{-\frac{\partial f}{\partial t}\right\}, \tag{59}$$

$$\{\dot{\mathbf{q}}\} = \begin{bmatrix}\dot{X}_P & \dot{Y}_P & \dot{Z}_P\end{bmatrix}^{\mathrm{T}}, \tag{60}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial t}\right) = & -\dot{X}_P \cos(\theta_0 \cos t)\sin t - X_P \sin^2 t \sin(\theta_0 \cos t) \\ & - X_P \cos t \cos(\theta_0 \cos t) + \dot{Y}_P \sin t \sin(\theta_0 \cos t) \\ & - Y_P \sin^2 t \cos(\theta_0 \cos t) + Y_P \cos t \sin(\theta_0 \cos t),\end{aligned} \tag{61}$$

$$\{\dot{\mathbf{C}}\} = -\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial t}\right)\right\}. \tag{62}$$

From the relation (9) we deduce the parameter λ, while the relation (10) offers the force $\mathbf{F}$.

4. Conclusion

The problem of the equilibrium without friction of a particle on a mobile surface with bilateral constraints is a difficult one. The determination of the necessary force for a certain equilibrium position reduces at some derivations. The inverse problem, the determination of the equilibrium position maybe solved only by numerical methods in the general case. The equilibrium positions in the case of a mobile surface may be completely different comparing to those when the surface is maintained fixed.

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