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Motion without Friction of a Particle on a Mobile Surface with Bilateral Constraints

In this paper we present the equations of motion of a particle in a multi-body type form on a mobile surface. The particle is considered to be acted by a resultant force. At this resultant, the normal reaction is also added. The normal reaction is variable in time, even if the particle remains at the same point of the mobile surface. An example is also presented.

Keywords: *mobile surface, equations of motion, multi-body, constraints*

1. Introduction

The problem of the motion of a particle on a surface is an old and still studied one. The dynamics of the particle on a fixed surface is completely solved [1] (analytical if it is possible, or, in the most general case, by numerical methods).

A general solution for the case of the rigid solid is given in [2], using a multibody approach and a matrix form for the equations of motion obtained from the general theorems and Lagrange's equations. The obtained equations are very complicate and their integration is difficult because the matrix of inertia can be not invertible.

Udwadia and Kalaba [3] use the Gauss principle and the Moore-Penrose inverse of the matrix to write the equations of motion for a system of particles and apply the theory to simple cases. A discussion of the use of this inverse of matrices in our case is given in the paper.

Let us consider a particle of mass m situated on the mobile surface of implicit equation

$$f(X, Y, Z, t) = 0, \tag{1}$$

in which t is the time.

The particle is acted by the force $\mathbf{F}$, which may be written as

$$\mathbf{F} = F_X\mathbf{i} + F_Y\mathbf{j} + F_Z\mathbf{k}, \tag{2}$$

where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the unit vectors of the axes OX, OY, and OZ, respectively.

In addition, $\mathbf{F}$ has the most general form, that is,

$$\mathbf{F} = \mathbf{F}\left(X, Y, Z, \dot{X}, \dot{Y}, \dot{Z}, t\right). \tag{3}$$

2. Equations of motion

These equations read

$$\begin{cases} m\ddot{X} = F_X + R_X , \\ m\ddot{Y} = F_Y + R_Y , \\ m\ddot{Z} = F_Z + R_Z , \end{cases} \tag{4}$$

in which R_X, R_Y, and R_Z are the components of the normal reaction.

On the other hand,

$$R_X = \lambda \frac{\partial f}{\partial X},\ R_Y = \lambda \frac{\partial f}{\partial Y},\ R_Z = \lambda \frac{\partial f}{\partial Z} \tag{5}$$

where λ is a real parameter (Lagrange's multiplier).

Deriving the relation (1) with respect to time, we obtain

$$\frac{\partial f}{\partial X}\dot{X} + \frac{\partial f}{\partial Y}\dot{Y} + \frac{\partial f}{\partial Z}\dot{Z} + \frac{\partial f}{\partial t} = 0, \tag{6}$$

which may be written as

$$[\mathbf{B}]\{\dot{\mathbf{q}}\} = \{\mathbf{C}\}, \tag{7}$$

where

$$[\mathbf{B}] = \begin{bmatrix} \frac{\partial f}{\partial X} & \frac{\partial f}{\partial Y} & \frac{\partial f}{\partial Z} \end{bmatrix}, \tag{8}$$

$$\{\mathbf{q}\} = [X\ \ Y\ \ Z]^{\mathrm{T}}, \tag{9}$$

$$\{\mathbf{C}\} = \left\{-\frac{\partial f}{\partial t}\right\}. \tag{10}$$

The system of the equations of motion becomes

$$m\ddot{X} = F_X + \lambda \frac{\partial f}{\partial X},\ m\ddot{Y} = F_Y + \lambda \frac{\partial f}{\partial Y}\ \text{m}\ m\ddot{Z} = F_Z + \lambda \frac{\partial f}{\partial Z} \tag{11}$$

at which one has to add the relation (6) or (7).

3. Matrix form of the equations of motion

Denoting

$$[\mathbf{m}] = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}, \tag{12}$$

$$\{\mathbf{F}\} = [F_X \;\; F_Y \;\; F_Z]^{\mathrm{T}}, \tag{13}$$

and deriving the equation (7) with respect to time,

$$[\dot{\mathbf{B}}]\{\dot{\mathbf{q}}\} + [\mathbf{B}]\{\ddot{\mathbf{q}}\} = \{\dot{\mathbf{C}}\}, \tag{14}$$

one obtains the matrix form of the equation of motion

$$\begin{bmatrix} [\mathbf{m}] & -[\mathbf{B}]^{\mathrm{T}} \\ [\mathbf{B}] & 0 \end{bmatrix} \begin{bmatrix} \{\ddot{\mathbf{q}}\} \\ \lambda \end{bmatrix} = \begin{bmatrix} \{\mathbf{F}\} \\ \{\dot{\mathbf{C}}\} - [\dot{\mathbf{B}}]\{\dot{\mathbf{q}}\} \end{bmatrix}. \tag{15}$$

We remark that

$$[\dot{\mathbf{B}}] = [\dot{b}_1 \;\; \dot{b}_2 \;\; \dot{b}_3], \tag{16}$$

$$\{\dot{\mathbf{C}}\} = \{\dot{c}\}, \tag{17}$$

where

$$b_1 = \frac{\partial f}{\partial X} \quad b_2 = \frac{\partial f}{\partial Y} \quad b_3 = \frac{\partial f}{\partial Z} \tag{18}$$

$$\dot{b}_1 = \frac{\partial^2 f}{\partial X^2}\dot{X} + \frac{\partial^2 f}{\partial X \partial Y}\dot{Y} + \frac{\partial^2 f}{\partial X \partial Z}\dot{Z} + \frac{\partial^2 f}{\partial X \partial t}, \tag{19}$$

$$\dot{b}_2 = \frac{\partial^2 f}{\partial Y \partial X}\dot{X} + \frac{\partial^2 f}{\partial Y^2}\dot{Y} + \frac{\partial^2 f}{\partial Y \partial Z}\dot{Z} + \frac{\partial^2 f}{\partial Y \partial t}, \tag{20}$$

$$\dot{b}_3 = \frac{\partial^2 f}{\partial Z \partial X}\dot{X} + \frac{\partial^2 f}{\partial Z \partial Y}\dot{Y} + \frac{\partial^2 f}{\partial Z^2}\dot{Z} + \frac{\partial^2 f}{\partial Z \partial t}, \tag{21}$$

$$c = -\frac{\partial f}{\partial t}, \tag{22}$$

$$\dot{c} = -\frac{\partial^2 f}{\partial t \partial X}\dot{X} - \frac{\partial^2 f}{\partial t \partial Y}\dot{Y} - \frac{\partial^2 f}{\partial t \partial Z}\dot{Z} - \frac{\partial^2 f}{\partial t^2}. \tag{23}$$

The equation (15) is a multi-body type one.

4. Solution of the matrix equations of motion

The equation (15) may be written as

$$\begin{cases} [\mathbf{m}]\{\ddot{\mathbf{q}}\} - [\mathbf{B}]^{\mathrm{T}}\lambda = \{\mathbf{F}\}, \\ [\mathbf{B}]\{\ddot{\mathbf{q}}\} = \{\dot{\mathbf{C}}\} - [\dot{\mathbf{B}}]\{\dot{\mathbf{q}}\}. \end{cases} \tag{24}$$

Multiplying the first relation (24) by $[\mathbf{m}]^{-1}$, we get

$$\{\ddot{\mathbf{q}}\} = [\mathbf{m}]^{-1}\{\mathbf{F}\} + [\mathbf{m}]^{-1}[\mathbf{B}]^{\mathrm{T}}\lambda. \tag{25}$$

Replacing the relation (25) in the second expression (24), one obtains

$$[\mathbf{B}]\{[\mathbf{m}]^{-1}\{\mathbf{F}\} + [\mathbf{m}]^{-1}[\mathbf{B}]^{\mathrm{T}}\lambda\} = \{\dot{\mathbf{C}}\} - [\dot{\mathbf{B}}]\{\dot{\mathbf{q}}\} \tag{26}$$

or

$$[\mathbf{B}][\mathbf{m}]^{-1}[\mathbf{B}]^{\mathrm{T}}\lambda = \{\dot{\mathbf{C}}\} - [\dot{\mathbf{B}}]\{\dot{\mathbf{q}}\} - [\mathbf{B}][\mathbf{m}]^{-1}\{\mathbf{F}\}. \tag{27}$$

Remembering now a well known result from the linear algebra, which states that if $[\mathbf{P}]$ is a full rank matrix and $[\mathbf{Q}]$ is an invertible matrix, then the product matrix $[\mathbf{P}][\mathbf{Q}][\mathbf{P}]^{\mathrm{T}}$ is invertible, it results that the matrix $[\mathbf{B}][\mathbf{m}]^{-1}[\mathbf{B}]^{\mathrm{T}}$ is invertible and the equation (27) offers the parameter λ,

$$\lambda = [[\mathbf{B}][\mathbf{m}]^{-1}[\mathbf{B}]^{\mathrm{T}}]\{\{\dot{\mathbf{C}}\} - [\dot{\mathbf{B}}]\{\dot{\mathbf{q}}\} - [\mathbf{B}][\mathbf{m}]^{-1}\{\mathbf{F}\}\}. \tag{28}$$

All we have to do now is to introduce the relation (28) in the first relation (24), resulting the equation

$$[\mathbf{m}]\{\ddot{\mathbf{q}}\} = \{\mathbf{F}\} + [\mathbf{B}]^{\mathrm{T}}\{[[\mathbf{B}][\mathbf{m}]^{-1}[\mathbf{B}]^{\mathrm{T}}]\{\{\dot{\mathbf{C}}\} - [\dot{\mathbf{B}}]\{\dot{\mathbf{q}}\} - [\mathbf{B}][\mathbf{m}]^{-1}\{\mathbf{F}\}\}\}. \tag{29}$$

This approach is similar to that given in [2].

Udwadia and Kalaba [3] use the Moore-Penrose inverse of a matrix. The system (24) may be put in the form

$$\begin{cases} \{\ddot{\mathbf{q}}\} = [\mathbf{m}]^{-1}\{\mathbf{F}\} + [\mathbf{m}]^{-1}[\mathbf{B}]^{\mathrm{T}}\lambda, \\ \{\ddot{\mathbf{q}}\} = [\mathbf{B}]^{+}\{\dot{\mathbf{C}}\} - [\mathbf{B}]^{+}[\dot{\mathbf{B}}]\{\dot{\mathbf{q}}\}, \end{cases} \tag{30}$$

where the superior index + signifies the Moore-Penrose inverse.

The system (30) leads to

$$[\mathbf{m}]^{-1}\{\mathbf{F}\} + [\mathbf{m}]^{-1}[\mathbf{B}]^{\mathrm{T}}\lambda = [\mathbf{B}]^{+}\{\dot{\mathbf{C}}\} - [\mathbf{B}]^{+}[\dot{\mathbf{B}}]\{\dot{\mathbf{q}}\}, \tag{31}$$

wherefrom

$$[\mathbf{m}]^{-1}[\mathbf{B}]^{\mathrm{T}}\lambda = [\mathbf{B}]^{+}\{\dot{\mathbf{C}}\} - [\mathbf{B}]^{+}[\dot{\mathbf{B}}]\{\dot{\mathbf{q}}\} - [\mathbf{m}]^{-1}\{\mathbf{F}\}; \tag{32}$$

hence

$$\lambda = [[\mathbf{m}]^{-1}[\mathbf{B}]^{\mathrm{T}}]^{+}\{[\mathbf{B}]^{+}\{\dot{\mathbf{C}}\} - [\mathbf{B}]^{+}[\dot{\mathbf{B}}]\{\dot{\mathbf{q}}\} - [\mathbf{m}]^{-1}\{\mathbf{F}\}\}. \tag{33}$$

Obviously, the relations (28) and (33) are one and the same.

5. Example

We consider that the particle moves on the mobile sphere of equation

$$f(X, Y, Z, t) = (X - a\cos\omega t)^2 + Y^2 + Z^2 - R^2 = 0, \tag{34}$$

and it is acted only by its own weight.

We successively obtain

$$[\mathbf{B}] = [2(X - a\cos\omega t)\ \ 2Y\ \ 2Z], \tag{35}$$

$$\{\mathbf{C}\} = \{2(X - a\cos\omega t)a\omega\sin\omega t\}, \tag{36}$$

$$[\dot{\mathbf{B}}] = [2(\dot{X} + a\omega\sin\omega t)\ \ 2\dot{Y}\ \ 2\dot{Z}], \tag{37}$$

$$\{\mathbf{q}\} = [X \;\; Y \;\; Z]^{\mathrm{T}}, \tag{38}$$

$$\{\mathbf{F}\} = [0 \;\; 0 \;\; -mg]^{\mathrm{T}}, \tag{39}$$

$$[\mathbf{m}] = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}, \tag{40}$$

$$[\mathbf{m}]^{-1} = \frac{1}{m}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{41}$$

$$[\mathbf{B}][\mathbf{m}]^{-1}[\mathbf{B}]^{\mathrm{T}} = \frac{4R^2}{m}, \tag{42}$$

$$[\dot{\mathbf{B}}]\{\dot{\mathbf{q}}\} = 2\dot{X}\left(\dot{X} + a\omega\sin\omega t\right) + 2\dot{Y}^2 + 2\dot{Z}^2, \tag{43}$$

$$[\mathbf{B}][\mathbf{m}]^{-1}\{\mathbf{F}\} = -2gZ, \tag{44}$$

$$\{\dot{\mathbf{C}}\} = 2(\dot{X} + a\omega\sin\omega t)a\omega\sin\omega t + 2(X - a\cos\omega t)a\omega^2\cos\omega t, \tag{45}$$

$$\lambda = \frac{m}{4R^2}\Big[2(\dot{X} + a\omega\sin\omega t)a\omega\sin\omega t + 2(X - a\cos\omega t)a\omega^2\cos\omega t - 2\dot{X}\left(\dot{X} + a\omega\sin\omega t\right) - 2\dot{Y}^2 - 2\dot{Z}^2 + 2gZ\Big], \tag{46}$$

$$[\mathbf{m}]^{-1}\{\mathbf{F}\} = [0 \;\; 0 \;\; -g]^{\mathrm{T}}, \tag{47}$$

$$\{\ddot{\mathbf{q}}\} = \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} + \frac{\lambda}{m}\begin{bmatrix} 2(X - a\cos\omega t) \\ 2Y \\ 2Z \end{bmatrix}. \tag{48}$$

The initial conditions must satisfy the relations

$$f\left(X^0, Y^0, Z^0, 0\right) = 0, \tag{49}$$

$$[\mathbf{B}]_{t=0}\begin{bmatrix} \dot{X}^0 \\ \dot{Y}^0 \\ \dot{Z}^0 \end{bmatrix} = \{\mathbf{C}\}|_{t=0}. \tag{50}$$

For instance, choosing $a = R$, at $t = 0$ we have

$$f\left(X, Y, Z.0\right) = Y^2 + Z^2 - R^2 = 0, \tag{51}$$

and we may select $X^0 = 0$, $Y^0 = 0$, $Z^0 = R$.

Moreover,

$$[\mathbf{B}]_{t=0} = [0 \;\; 0 \;\; 2R], \tag{52}$$

$$\{\mathbf{C}\}|_{t=0} = \{0\}, \tag{53}$$

and the relation (50) becomes

$$2R\dot{Z}^0 = 0. \quad (54)$$

We may choose $\dot{X}^0 = 0$, $\dot{Y}^0 = 0$, $\dot{Z}^0 = 0$.

With these initial conditions, the problem becomes a planar one (the motion takes place in the plan OXZ) and it can be solved by elementary methods.

6. Conclusion

In this paper we studied the motion without friction of a particle on a mobile surface in the most general case. The equations of motion are deduced and put in a matrix form easily used in any application. Two variants for the integration of this matrix equation are given. The first one bases in the fact that the matrix of inertia is an invertible one, and the second one uses the Moore-Penrose inverse of a matrix. An example clarifies the theory.

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