ANALELE UNIVERSITĂȚII
"EFTIMIE MURGU" REȘIȚA
ANUL XVIII, NR. 2, 2011, ISSN 1453 - 7397

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Oversampling Operators: Frame Representation of Operators

For a numerical solution of operator equations a discretization of the operators is necessary. In the well-known Boundary Element Method (BEM) the Galerkin approach uses bases to do that. Frames are often easier or faster to construct than bases. Here we look at the matrix representation of operators using frames. We give a survey over the basic results, with particular focus on the invertibility of the involved systems.

Keywords*: Frames, matrix representation, discretization of operators, operator equations.*

1. Introduction

The solution of operator equation is ubiquitous in applied mathematics. In computational acoustics, for example, one aims to solve operator equations numerically, such as equations for vibration or sound field analysis. Here the finite element [15] and the boundary element method [20] are widely used. One particular scheme to discretize the operator equations is the Galerkin method [13]. This corresponds to taking finite sections of the standard matrix description [14] of operators O using an ONB (or biorthogonal basis) (e_k) by constructing a matrix M with the entries

$$M_{j,k} = < Oe_k, e_j > .$$

But the search for bases with certain properties, like sparsity of the system matrix, can be a very restrictive approach. The relaxation and generalization to frames [9, 11] can lead to more stable and faster algorithms. Recently the representation of operators using frames has received some attention [1, 2, 7, 19]. Certain operators, named multipliers, which have a diagonal representation, are of special interest in mathematics [3, 5, 6] as well as acoustical applications [4, 12, 17, 18].

In this paper we give a survey over results about the representation of operators using frames, in particular in connection with the invertibility of operators and the connected matrices. For proofs we refer to [7] and [8].

2. Preliminaries and Notation

We largely stick to the notation in [7], let us just remind the reader on the concept of frames.

2.1. Frames

For more details and proofs for this section refer e.g. to [10, 9].

A sequence $\Psi = (\psi_k \mid k \in K)$ is called a *frame* for the Hilbert space $\mathcal{H}$, if constants $A, B > 0$ exist, such that

$$A \cdot \|f\|_{\mathcal{H}}^2 \leq \sum_k |< f, \psi_k >|^2 \leq B \cdot \|f\|_{\mathcal{H}}^2 \;\forall\; f \in \mathcal{H} \qquad (1)$$

Here A is called a *lower* and B an *upper frame bound .*

A sequence $\Psi = (\psi_k)$ is called a *Bessel sequence* with Bessel bound B if it fulfills the right inequality above.

For a Bessel sequence $\Psi = (\psi_k)$ let $C_\Psi : \mathcal{H} \to l^2(K)$ be the *analysis operator* $C_\Psi(f) = (< f, \psi_k >)_k$. Let $D_\Psi : l^2(K) \to \mathcal{H}$ be the *synthesis operator* $D_\Psi((c_k)) = \sum_k c_k \cdot \psi_k$. Let $S_\Psi : \mathcal{H} \to \mathcal{H}$ be the (associated) *frame operator* $S_\Psi(f) = \sum_k < f, \psi_k > \cdot \psi_k$.

For a frame $\Psi = (\psi_k)$ with bounds A, B, the operator C is a bounded, injective operator with closed range and $S = C^* C = D D^*$ is a positive invertible operator satisfying $A I_{\mathcal{H}} \leq S \leq B I_{\mathcal{H}}$ and $B^{-1} I_{\mathcal{H}} \leq S^{-1} \leq A^{-1} I_{\mathcal{H}}$, where $I_{\mathcal{H}}$ denotes the identity on $\mathcal{H}$. Even more, we can find an expansion for every member of $\mathcal{H}$: The sequence $\tilde{\Psi} = (\tilde{\psi}_k) = (S^{-1} \psi_k)$ is a frame with frame bounds B^{-1}, $A^{-1} > 0$, the so called *canonical dual frame .* Every $f \in \mathcal{H}$ has the expansions $f = \sum_{k \in K} < f, \tilde{\psi}_k > \psi_k$ and $f = \sum_{k \in K} < f, \psi_k > \tilde{\psi}_k$ where both sums converge unconditionally in $\mathcal{H}$.

2.2 Motivation: Solving Operator Equalities

2.2.1 A Typical Operator Equation in Acoustics

The boundary element method (BEM) is a widely used numerical method to solve radiation and scattering problems [20]. Compared with finite element methods (FEM) [15], it has the advantage that only the surfaces of the radiating/reflecting objects have to be discretized with a mesh, but not the objects themselves or the space surrounding them, which is especially advantageous for problems in unbounded domains (exterior problems).

The problem of scattering and radiation of acoustic waves from an object Ω with an acoustically hard reflecting (closed) Lipschitz surface Γ in an unbounded domain Ω^e exterior to that object are described by the Helmholtz equation. Using the fundamental solution of the Helmholtz equation $G(\mathbf{x},\mathbf{y}) = \frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}$, it can be given by the integral equation

$$u(\mathbf{x}) = -\int_\Gamma G(\mathbf{x},\mathbf{y}) \frac{\partial u(\mathbf{y})}{\partial \mathbf{n}_\mathbf{y}} ds_y + \int_\Gamma \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}_\mathbf{y}} u(\mathbf{y}) ds_y \tag{2}$$

$$= -(Sv)(\mathbf{x}) + (Du)(\mathbf{x}), \quad \mathbf{x} \in \Omega^e \tag{3}$$

where $\mathbf{n}_\mathbf{y}$ denotes the normal vector at the point $\mathbf{y}$ and $v(\mathbf{y}) = \frac{\partial u(\mathbf{y})}{\partial \mathbf{n}_\mathbf{y}}$.

A commonly used method to transform Eq. (2) into a linear system of equations is the above-mentioned Galerkin discretization. The whole system is projected onto a finite-dimensional space $V_N = \text{span}\{\theta_i(\mathbf{x}), i = 1 \ldots N\}$, where the θ_i form an orthogonal basis. So a finite-dimensional matrix is built for S by using $M^S_{j,k} =< S\theta_k, \theta_j >$ and in an analogue way for D.

The BEM can be, for example, be used for a modeling of human sound loclaization. The shape of the human head, torso, and pinna play an important role in localization of sounds. The reflections, especially at the pinna, act as a filter, which can be described by the head-related transfer functions. This filters can be estimated from a 3D-scan of the head, using BEM. However, it is not possible to use the BEM directly with such fine meshes because of memory and computation limits. Therefore, recently matrix compression techniques like the fast multipole method have been used to calculate HRTFs for high frequencies [16].

2.2.2 Frame Approach

Given an operator equation

$$O \cdot f = g \tag{4}$$

we want to discretize it to find a numerical solution using a frame $\Phi = (\phi_k)$. For a given g with coefficients $d = (d_k) = (< g, \phi_k >)$ and a matrix representation M of O there are several algorithms to find the least square solution of

$$M \cdot c = d. \tag{5}$$

To find a true solution for Eq. 4 we can now apply $D_{\tilde{\Phi}}$ on c. Although, in general c is not in $ran(C_\Phi)$ even if d is, this is not a problem. Rephrasing Eq. 4 we see the following:

$$Of = g \Leftrightarrow \sum_k < f, \phi_k > O\tilde{\phi}_k = g \Leftrightarrow \sum_k < f, \phi_k >< O\tilde{\phi}_k, \phi_l >=< g, \phi_l >.$$

This gives us an algorithm for finding an approximative solution to the inverse operator problem $Of = g$.

1. Define the matrix M by $M_{k,l} =< O\tilde{\phi}_k, \phi_l >$.
2. Find a good finite dimensional approximation M_N of M by using the finite section method [14].
3. Apply an algorithm like e.g. the QR factorization [21] to find a solution for Eq. 5.
4. Synthezise with the dual frame $\tilde{\Phi}$.

3 Matrix Representation

We will start with the more general case of Bessel sequences. Note that we will use the notation $\|\cdot\|_{\mathrm{H}_1 \to \mathrm{H}_2}$ for the operator norm in $\mathrm{B}(\mathrm{H}_1, \mathrm{H}_2)$, i.e. the space of bounded operators from H_1 into H_2, to be able to distinguish between different operator norms.

Theorem 3.1 *Let* $\Psi = (\psi_k)$ *be a Bessel sequence in* H_1 *with bound* B*,* $\Phi = (\phi_k)$ *in* H_2 *with* B'*.*

1. Let $O : \mathrm{H}_1 \to \mathrm{H}_2$ be a bounded, linear operator. Then the infinite matrix

$$\left(\mathsf{M}^{(\Phi,\Psi)}(O)\right)_{m,n} = < O\psi_n, \phi_m >$$

defines a bounded operator from l^2 to l^2 with $\|\mathsf{M}\|_{l^2 \to l^2} \le \sqrt{B \cdot B'} \cdot \|O\|_{\mathsf{H}_1 \to \mathsf{H}_2}$. As an operator $l^2 \to l^2$

$$\mathsf{M}^{(\Phi,\Psi)}(O) = C_\Phi \circ O \circ D_\Psi .$$

This means the function $\mathsf{M}^{(\Phi,\Psi)} : \mathsf{B}(\mathsf{H}_1, \mathsf{H}_2) \to \mathsf{B}(l^2, l^2)$ is a well-defined bounded operator.

2. On the other hand let M be an infinite matrix defining a bounded operator from l^2 to l^2, $(Mc)_i = \sum_k M_{i,k} c_k$. Then the operator $\mathsf{O}^{(\Phi,\Psi)}$ defined by

$$\left(\mathsf{O}^{(\Phi,\Psi)}(M)\right)h = \sum_k \left(\sum_j M_{k,j} < h, \psi_j > \right) \phi_k, \ for \ h \in \mathsf{H}_1$$

is a bounded operator from H_1 to H_2 with

$$\left\| \mathsf{O}^{(\Phi,\Psi)}(M) \right\|_{\mathsf{H}_1 \to \mathsf{H}_2} \le \sqrt{B \cdot B'} \|M\|_{l^2 \to l^2} .$$

$$\mathsf{O}^{(\Phi,\Psi)}(M) = D_\Phi \circ M \circ C_\Psi = \sum_k \sum_j M_{k,j} \cdot \phi_k \otimes_i \overline{\psi}_j$$

This means the function $\mathsf{O}^{(\Phi,\Psi)} : \mathsf{B}(l^2, l^2) \to \mathsf{B}(\mathsf{H}_1, \mathsf{H}_2)$ is a well-defined bounded operator.

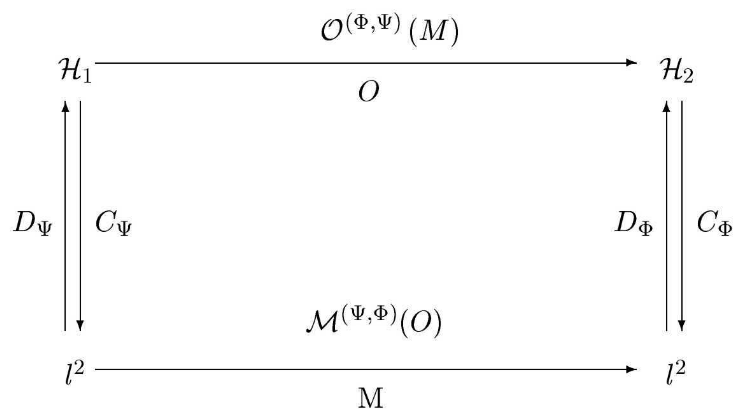

Figure 1: The operator induced by a matrix *M* and the matrix induced by an operator *O*.

Definition 3.1 *For an operator* O *and a matrix* M *as in Theorem 3.1, we call* $\mathcal{M}^{(\Psi,\Phi)}(O)$ *the matrix induced by the operator* O *with respect to the Bessel sequences* $\Psi=(\psi_k)$ *and* $\Phi=(\phi_k)$ *and* $\mathcal{O}^{(\Psi,\Phi)}(M)$ *the operator induced by the matrix* M *with respect to the Bessel sequences* Ψ *and* Φ. *(See Figure 1.)*

For frames we can prove more properties:

Proposition 3.2 *Let* $\Psi=(\psi_k)$ *be a frame in* $\mathcal{H}_1$ *with bounds* A,B, $\Phi=(\phi_k)$ *in* $\mathcal{H}_2$ *with* A',B'. *Then*

1. $\left(\mathcal{O}^{(\Phi,\Psi)}\circ M^{(\tilde{\Phi},\tilde{\Psi})}\right)=I_{B(H_1,H_2)}=\left(\mathcal{O}^{(\tilde{\Phi},\tilde{\Psi})}\circ M^{(\Phi,\Psi)}\right)$.

   And therefore for all $O\in\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)$:

   $$O=\sum_{k,j}<O\tilde{\psi}_j,\tilde{\phi}_k>\phi_k\otimes_i\overline{\psi}_j$$

2. $\mathcal{M}^{(\Phi,\Psi)}$ is injective and $\mathcal{O}^{(\Phi,\Psi)}$ is surjective.
3. Let $\mathcal{H}_1=\mathcal{H}_2$, then $\mathcal{O}^{(\Psi,\tilde{\Psi})}(Id_{l^2})=Id_{\mathcal{H}_1}$
4. Let $\Xi=(\xi_k)$ be any frame in $\mathcal{H}_3$, and $O:\mathcal{H}_3\to\mathcal{H}_2$ and $P:\mathcal{H}_1\to\mathcal{H}_3$. Then

   $$\mathcal{M}^{(\Phi,\Psi)}(O\circ P)=\left(\mathcal{M}^{(\Phi,\Xi)}(O)\cdot\mathcal{M}^{(\tilde{\Xi},\Psi)}(P)\right).$$

We can show some more connections of operators and their associated matrices:

Proposition 3.3 *Let* Φ *and* Ψ *be frames for* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *respectively, and* $G_{\Psi,\Phi}=C_\Psi\circ D_\Phi$ *be the Gram matrix. Then the following are equivalent:*

1. $\exists O\in\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)$ such that $M=\mathcal{M}(O)$
2. $\exists M'\in\mathcal{B}(\ell_2,\ell_2)$ such that $M=\mathcal{M}(\tilde{\mathcal{O}}(M'))$
3. $\operatorname{ran}(M)\subseteq\operatorname{ran}(C_\Phi)$ and $\ker(D_\Psi)\subseteq\ker(M)$
4. $G_{\Phi,\tilde{\Phi}}\circ M\circ G_{\Psi,\tilde{\Psi}}=M$

4 Invertibility

In particular for solving operator or matrix equations the invertibility of the involved systems is of interest. We can show:

Lemma 4.1 *Let* $M\in\mathcal{B}(\ell_2,\ell_2)$.

1. If $\Pi_{\mathrm{ran}(C_\Phi)} \circ M$ injective (on $\mathrm{ran}(C_\Psi)$), then $\mathcal{O}^{(\Phi,\Psi)}(M)$ is injective.

2. If $\Pi_{\mathrm{ran}(C_\Phi)} \circ M |_{\mathrm{ran}(C_\Psi)}$ is surjective onto $\mathrm{ran}(C_\Phi)$, then $\mathcal{O}^{(\Phi,\Psi)}(M)$ is surjective.

3. If O is bijective, then $M = \mathcal{M}^{(\Phi,\Psi)}(O)$ is bijective as operator from $\mathrm{ran}(C_\Psi)$ onto $\mathrm{ran}(C_\Phi)$.

4. If $\Pi_{\mathrm{ran}(C_\Phi)} \circ M$ is bijective as an operator from $\mathrm{ran}(C_\Psi)$ onto $\mathrm{ran}(C_\Phi)$, then $\mathcal{O}^{(\Phi,\Psi)}(M)$ is bijective.

In particular this means that we can express the relation of the inverses of associated matrices and operators:

Theorem 4.2 *Let* O *be invertible. Then* $M = \mathcal{M}(O)$ *is invertible and* $M^{-1} = \mathcal{M}^{(\tilde{\Psi},\tilde{\Phi})}(O^{-1}) = G_{\tilde{\Psi},\tilde{\Phi}} \circ \mathcal{M}^{(\Phi,\Psi)}(O^{-1}) \circ G_{\tilde{\Psi},\tilde{\Phi}}$ *.*

Let $M : \mathrm{ran}(C_\Psi) \to \mathrm{ran}(C_\Phi)$ be invertible. Then $O = \mathcal{O}^{\Phi,\Psi}(M)$ is invertible and $O^{-1} = \mathcal{O}^{(\tilde{\Phi},\tilde{\Psi})}(M^{-1}) = \mathcal{O}^{\Phi,\Psi}(G_{\tilde{\Psi},\Psi} M^{-1} G_{\Phi,\tilde{\Phi}})$.

5 Summary and Outlook

We have shown some basic properties of frame representations of operators, in particular with regard to their invertibility.

In the future, we are planning to investigate the relation of the operator representation using frames presented here with special focus on the finite section method and localized frames. Furthermore we will apply this concept to the numerical solution of the Helmholtz equation using wavelet frames.

Acknowledgments The authors are thankful to W. Kreuzer and J.-P. Antoine for valuable comments and discussions.

References

[1] Ambroziski Z., Rudolf K., *Matrices defined by frames*, Opuscula Math. 29 (2009), no. 4, 365–375.

[2] Arias M.L., Corach G., Pacheco M., *Characterization of Bessel sequences*, Extracta mathematicae 22 (2007), no. 1, 55–66.

[3] Arias M.L., Pacheco M., *Bessel fusion multipliers*, Journal of Mathematical Analysis and Applications 348 (2008), no. 2, 581 – 588.

[4] Balazs P., Laback B, Eckel G., Deutsch W.A., *Time-frequency sparsity by removing perceptually irrelevant components using a simple model of simultaneous masking*, IEEE Transactions on Audio, Speech and Language Processing 18 (2010), no. 1, 34–49.
[5] Balazs P., *Basic definition and properties of Bessel multipliers*, Journal of Mathematical Analysis and Applications 325 (2007), no. 1, 571–585.
[6] Casazza P., *The art of frame theory*, Taiwanese J. Math. 4 (2000), no. 2, 129–202.
[7] Christensen O., *An introduction to frames and Riesz bases*, Birkhäuser, 2003.
[8] Christensen O., *Frames and Bases. An Introductory Course*, Applied and Numerical Harmonic Analysis. Basel Birkhäuser, 2008.
[9] Depalle P., Kronland-Martinet R., Torrésani B., *Time-frequency multipliers for sound synthesis,* Proceedings of the Wavelet XII conference, SPIE annual Symposium, San Diego, August 2007.
[10] Gaul L., Kögler M., Wagner M., *Boundary element methods for engineers and scientists,* Springer, 2003.
[11] Gohberg I., Goldberg S., Kaashoek M.A., *Basic classes of linear operators*, Birkhäuser, 2003.
[12] Hackbusch W., *Elliptic differential equations. Theory and numerical treatment*, Springer-Verlag, Berlin, 2003.
[13] Kreuzer W., Majdak P., Chen Z., *Fast multipole boundary element method to calculate head-related transfer functions for a wide frequency range*, J. Acoust. Soc. Am. 126 (2009), no. 3, 1280–1290.
[14] Majdak P., Balazs P., Kreuzer W, Dörfler M., *A time-frequency method for increasing the signal-to-noise ratio in system identification with exponential sweeps*, Proceedings of the 36th International Conference on Acoustics, Speech and Signal Processing, ICASSP 2011, Prag, 2011.
[15] Olivero A., Torrésani B., Kronland-Martinet R., *A new method for Gabor multipliers estimation: Application to sound morphin*g., EUSIPCO-2010, 2010.

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