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Veturia Chiroiu, Petre P.Teodorescu, Ligia Munteanu

On the Geometric Transformations and Auxetic Materials

A new approach to obtain various architectures for auxetic foams by using the property of Helmholtz equation to be invariant under geometric transformations is described in this paper. The versatility of the geometric transformations is illustrated in order to obtain the auxetic version from the conventional foam.

***Keywords**: Geometric transformation, Auxetic foams, Cosserat theory, Chirality.*

1. Introduction

Since 1987, when isotropic auxetic foam was manufactured for the first time, negative Poisson’s ratio materials have created interest for potential engineering applications. A feature that the auxetic materials showed compared to the other foams is the significant damping capacity with increase up to 16 times compared to the conventional foam [1-5]. All the major classes of polymers, composites, metals and ceramics, can exist in the auxetic version [6]. Cellular solids are two phase composite materials in which one phase is a solid and the other is a fluid, most often air. The positive Poisson’s ratio is the result of the convex shape of cell surfaces. By volume compression, a part of the cell surface may acquire first a zeroth and then negative curvature. When the number of such inverted cells dominates, compressibility of the material rises till the appearance of the negative Poisson’s ratio [7]. Fig.1 shows the simplest 2D lattice structures for closed-cell solids: (a) square structure, (b) rectangle structure, (c) regular hexagonal structure, d) triangular structure and d) irregular structure. Fig.2 illustrates models of auxetic structures obtained from aforementioned materials by appearance of the negative Poisson’s ratio: a), b) re-entrant honeycomb network, c) Shilko and Konyok model [7] of the inverted closed-cell foam, and d) re-entrant regular array of rectangular nodules interconnected by fibrils [8].

The conventional foam exhibits pores with average diameter around 1mm while the auxetic foam has average diameter possible down to a few micrometers or down to a few nanometers.

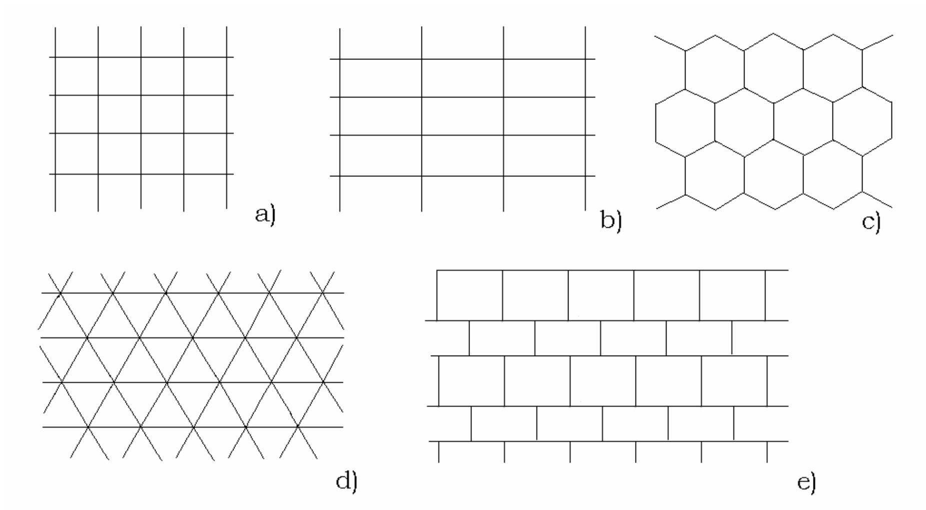

Figure 1. Regular 2D cellular solids.

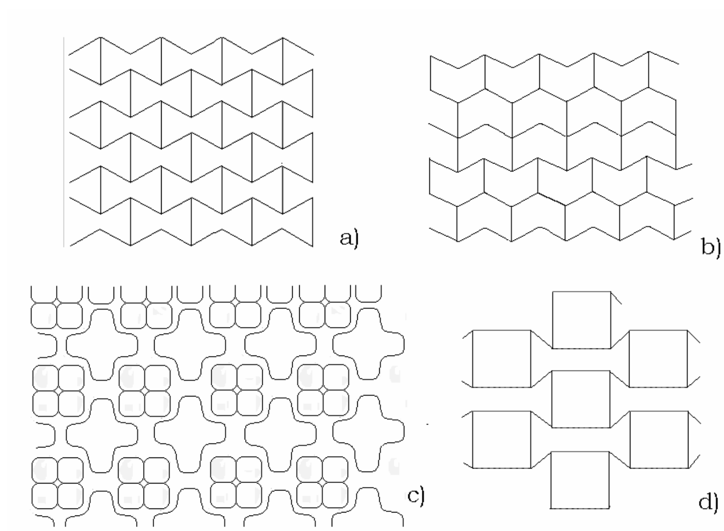

Figure 2. Model structures of the inverted closed-cell foam.

In this paper, a new technique for transforming the conventional foams into auxetic foams is proposed. The scope is to transform conventional foam which occupies the disk $r \le R_2$ into auxetic material which fills the annulus $R_1 \le r \le R_2$. The new material is inhomogeneous and anisotropic.

The geometric transformations cannot be applied to equations which are not invariant under coordinate transformations and, consequently, if cloaking exists for such equations (for example the elasticity equations), it would be of a different nature from acoustic and electromagnetic [9].

The touchstone of our technique is that the governing equations of the wave propagation through non-auxetic foams are invariant under geometric transformations, more precisely, the equations simplify into the Helmholtz equation [10]. So, the geometric transformations are used in order to obtain various inhomogeneous and anisotropic auxetic materials.

2. Geometric transfomations

A finite size object surrounded by a coating consisting of a specially designed metamaterial would become invisible for electromagnetic waves at any frequency [9]. The idea is that the sound sees the space differently [11]. For the sound, the concept of distance is modified by the acoustic properties of the regions through which the sound travels. In geometrical acoustics, the idea of the acoustical path when travelling an infinitesimal distance $\mathrm{d}s$, is the corresponding acoustical path length $c^{-1}\mathrm{d}s$, where $c^{-1} = \sqrt{\rho/\kappa}$ with ρ is the fluid density and κ is the compression modulus of the fluid. For example, the 3D equation for the pressure waves propagating in a bounded fluid region $\Omega \subset \mathrm{R}^3$ is the Helmholtz equation

$$\nabla \cdot (\underline{\underline{\rho}}^{-1} \nabla p) + \frac{\omega^2}{\kappa} p = 0 , \tag{1}$$

where p is the pressure, $\underline{\underline{\rho}}$ is the rank-2 tensor of the fluid density, κ is the compression modulus of the fluid, and ω is the wave frequency.

Geometric transformations applied to certain types of elastodynamic waves in structural mechanics received less attention, since the Navier equations do not usually retain their form under geometric changes [12]. For example, the in-plane propagation of time-harmonic elastic waves is governed by the Navier equations

$$\nabla \cdot C : \nabla U + \rho\omega^2 U + b = 0 , \tag{2}$$

where u is the displacement, ρ the density, C the 4th-order material tensor of the linear elastic material and $b(x)$ represents the spatial distribution of a simple harmonic body force $\hat{b}(x,t) = b(x)\exp(\mathrm{i}\omega t)$, with the wave-frequency and t the time. The Navier equations (2) retain their form under the transformation [10]

$$r' = r_0 + \frac{r - r_{01}}{r_1} r\,,\ \theta' = \theta \ \text{for} \ r \le r_1\,,\ r = r'\,,\ \theta = \theta'\,,\ \text{for} \ r > r_1\,, \tag{3}$$

where r_0 and r_1 are the inner and outer radii of the circular domain, respectively.

Let us consider the geometric transformation from the coordinate system (x', y', z') of the compressed space to the original coordinate system (x, y, z), given by $x(x', y', z')$, $y(x', y', z')$ and $z(x', y', z')$. The change of coordinates is characterized by the transformation of the differentials through the Jacobian $J_{xx'}$ of this transformation, i.e.

$$\begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{pmatrix} = J_{xx'} \begin{pmatrix} \mathrm{d}x' \\ \mathrm{d}y' \\ \mathrm{d}z' \end{pmatrix}, \quad J_{xx'} = \frac{\partial(x, y, z)}{\partial(x', y', z')}\,. \tag{4}$$

From the geometrical point of view, the change of coordinates implies that, in the transformed region, one can work with an associated metric tensor [13, 14]

$$T = \frac{J_{xx'}^{\mathsf{T}} J_{xx'}}{\det(J_{xx'})}\,. \tag{5}$$

In terms of the material parameters, one can replace the material from the original domain (homogeneous and isotropic) by an equivalent compressed one that is inhomogeneous (its characteristics depend on the spherical (r', θ', ϕ') coordinates) and anisotropic (described by a tensor), and whose properties, in terms of $J_{x'x}$, are given by

$$\underline{\underline{\rho}}' = J_{x'x}^{-\mathsf{T}} \cdot \rho \cdot J_{x'x}^{-1} \cdot \det(J_{x'x})\,, \quad \kappa' = \kappa \det(J_{x'x})\,, \tag{6}$$

or, equivalently, in terms of $J_{xx'}$

$$\underline{\underline{\rho}}' = \frac{J_{xx'}^{\mathsf{T}} \cdot \rho \cdot J_{x'x}}{\det(J_{xx'})}\,,\ \kappa' = \frac{\kappa}{\det(J_{xx'})}\,. \tag{7}$$

Here, $\underline{\underline{\rho}}'$ is a second order tensor. When the Jacobian matrix is diagonal, (6) and (7) can be more easily written. Multiplying (1) by a test function φ and integrating by parts, one obtains

$$-\int_{\Omega} \left(\nabla_{(x,y,z)} \varphi \cdot \underline{\underline{\rho}}^{-1} \nabla_{(x,y,z)} p \right) \mathrm{d}V + \int \left(\omega^2 \kappa^{-1} p \varphi \right) \mathrm{d}V = 0\,. \tag{8}$$

In (8) the surface integral, corresponding to a Neumann integral over the boundary $\partial\Omega$, is zero. By applying the coordinate transformation $(x, y, z) \to (x', y', z')$ to (8) and using (4), one obtains

$$-\int_{\Omega}\left(J_{x'x}^{\mathrm{T}}\nabla_{(x',y',z')}\varphi\cdot\underline{\underline{\rho}}^{-1}J_{x'x}^{\mathrm{T}}\nabla_{(x,y,z)}p\right)\det(J_{xx'})\mathrm{d}V'+\int\left(\det(J_{xx'})\omega^2\kappa^{-1}p\varphi\right)\mathrm{d}V'=0, \quad (9)$$

in terms of $J_{xx'}$, and

$$-\int_{\Omega}\left(\left(\nabla_{(x',y',z')}\varphi\right)^{\mathrm{T}}\frac{J_{x'x}\underline{\underline{\rho}}^{-1}J_{x'x}^{\mathrm{T}}}{\det(J_{x'x})}\nabla_{(x',y',z')}p\right)\mathrm{d}V'+\int\left(\frac{\kappa^{-1}}{\det(J_{x'x})}\omega^2 p\varphi\right)\mathrm{d}V'=0, \quad (10)$$

in terms of $J_{x'x}$.

3. Formulation of the problem

The geometric transformation may be linear or nonlinear. A linear geometric transformation (4) which maps the disk $r \le R_2$ into an annulus $R_1 \le r \le R_2$ is given by [9]

$$r' = R_1 + r\frac{R_2 - R_1}{R_2}, \quad 0 \le r \le R_2,$$

$$\theta' = \theta, \; 0 \le \theta \le 2\pi, \quad (11)$$

$$x_3' = x_3, \quad x_3 \in \mathrm{R},$$

where r', θ', x_3' are radially contracted cylindrical coordinates r, θ, x_3. The Cartesian basis (x_1, x_2, x_3) is defined as $x_1 = r\cos\theta$, $x_2 = r\sin\theta$. The Jacobean of the transformation from polar to stretched polar coordinates is given by $J_{rr'} = \dfrac{\partial(r,\theta,x_3)}{\partial(r',\theta',x_3')}$. In the stretched space, the associated metric tensor is given by (5)

$$T = \frac{J_{rr'}^{\mathrm{T}}\, J_{rr'}}{\det(J_{rr'})}. \quad (12)$$

Once the above geometric transforms are written, let us formulate the problem to be solved in this paper. Let us suppose that the original domain is a cylinder of radius R_2 and length l. This domain is filled with conventional non-auxetic cellular foam. The spatial compression is obtained by applying the geometric transformation (11). The transformed domain is a shell cylinder of internal and external radii R_1 and R_2, respectively, and the length l'.

We suppose that the material is a micropolar solid with chiral effects, i.e. a noncentrosymmetric material. The micropolar and classical theories of elasticity are continuum theories, which make no reference to atoms or other structural features of the material, which is described. Elasticity theory represents more than an analytical description of the phenomenological behavior since it can be derived as a first approximation of the interaction between atoms in the solid [16. 17].

4. Results

The original material is conventional closed-cell polyurethane foam with $\rho_{conv} = 27\,\mathrm{kg/m^3}$ density and Poisson's ratio at tensile test $\nu_{tens} = 0.47$ and the compressive test $\nu_{comp} = 0.27$. The cylindrical specimen has $R_2 = 15$mm initial radius and $l = 170$mm initial length. We must say that the condition of a positive Young's modulus and $-1 < \nu < 0.5$ corresponds to the usual range of properties for stability of the material. The existence of negative material constants (shear modulus, bulk modulus, stiffness) is also permitted [18,19].

The Poisson's ratio $\nu = \nu_{yx}$ (for tensile and compressive tests) was calculated as the negative ratio between the radial and longitudinal strains using a best fit to the strain-strain graph

$$\nu_{yx} = -\frac{\varepsilon_x}{\varepsilon_y}. \tag{13}$$

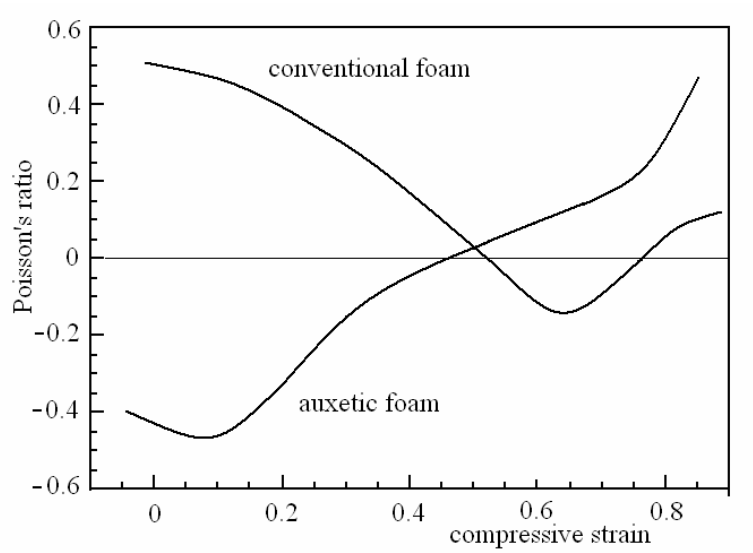

Figure 3. Poisson's ratio versus compressive strain for conventional and auxetic foams.

The most important physical parameter to dominate the negative Poisson's ratio transformation is the compression ratio $\vartheta = \dfrac{(R_2'^2 - R_1'^2)l'}{R_2^2 l}$, where prime denotes the final parameters.

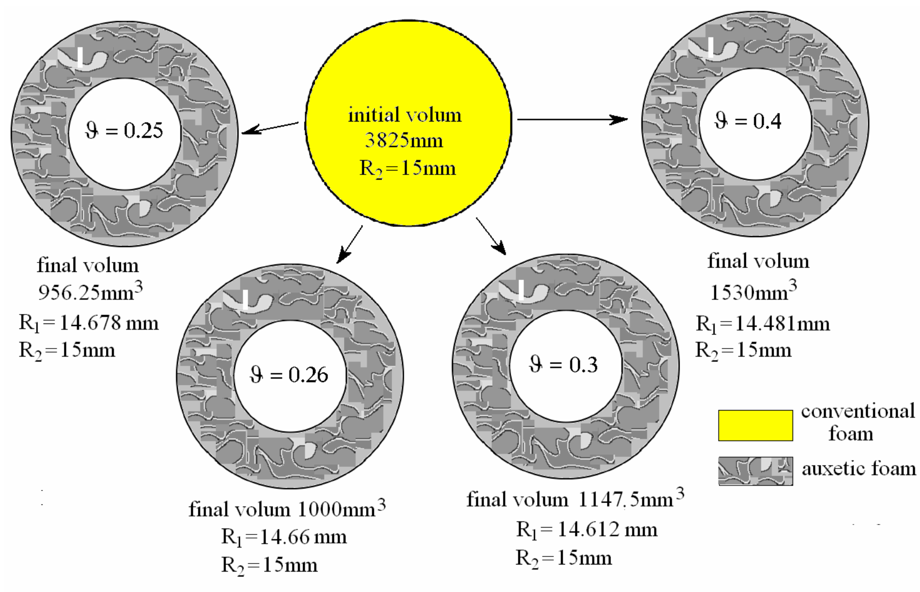

Figure 4. Transformed domains.

Fig.3 shows the variation of the Poisoon's ratio with respect to $1-\vartheta$ (equivalent to the compressive strain) for conventional foam (the upper curve) and auxetic foam (the lower curve) respectively. We observe that the conventional foam becomes auxetic ($-0.15 < \nu < 0$) for $0.55 < 1-\vartheta < 0.77$, or $0.23 < \vartheta < 0.45$. It is very interesting to see that the auxetic foam is changing the sign for its Poisson's ratio for $0.46 > 1-\vartheta$.

It is of interest to underline that the results provides an overall agreement with the experimental values for the auxetic foam [7, 20].

The initial domain with $R_2 = 15$mm and $l = 170$mm is transformed into a shell cylinder with $l' = 100$mm, $R_2' = 15$mm and 14.415mm$< R_1' <$14.7mm. The transformed annulus domains are presented in Fig. 4, for $\vartheta =$ 0.25, 0.26, 0.3 and 0.4.

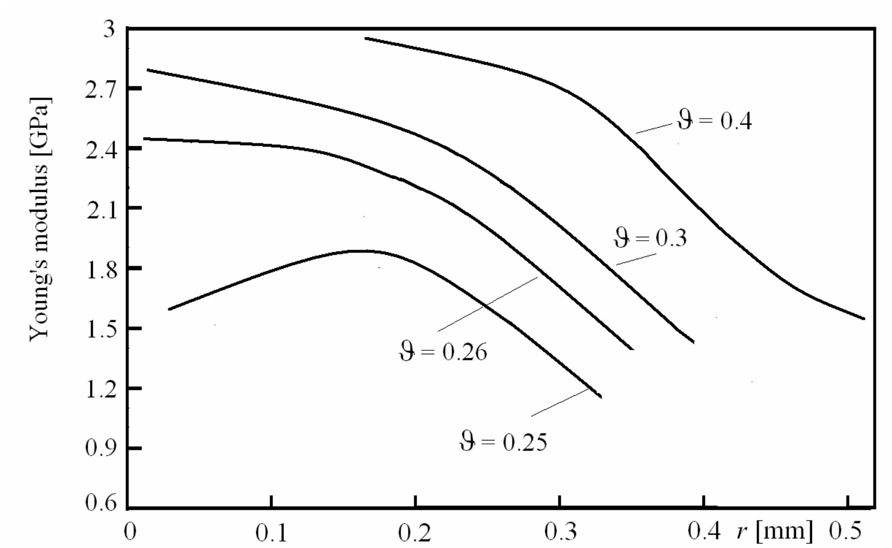

Figure 5. Variation of the Young's modulus with respect to radial coordinate.

The variation of the Young's modulus with respect to radial coordinate $R_1 \le r \le R_2$ is presented in Fig.5 for $\vartheta = 0.25$, 0.26, 0.3 and 0.4 (the corresponding thicknesses for the annulus $R_1 \le r \le R_2$ are 0.322mm, 0.34mm, 0.3888 and 0.519 respectively).

5. Conclusion

A new technique for transforming the conventional foams into auxetic foams is proposed in this paper by exploiting the property of the governing equations to be written in a covariant form such that the metric is only involved in the material parameters. The geometric transformations lead to material properties that are, if not impossible to obtain, at least challenging for the auxetic foams manufacture.

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Addresses:

- Dr. Veturia Chiroiu, Institute of Solid Mechanics, Romanian Academy, Ctin Mille 15, 010141 Bucharest, veturiachiroiu@yahoo.com
- Prof. Dr. Eng. Petre P.Teodoorescu, University of Bucharest, Mathematics Department, petre_teodorescu@hotmail.com
- Dr. Ligia Munteanu, Institute of Solid Mechanics, Romanian Academy, Ctin Mille 15, 010141 Bucharest, ligia_munteanu@hotmail.com