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Calculation Method for the Displacements of Cross Sections for Unsupported Straight Bars Subjected to Axial or Torsion Loads

This paper discusses a method for the determination of the displacements of the cross sections for unsupported straight bars subjected to axial loads – with forces or moments. The displacements of the cross sections are determined against their initial position, as well as against the section or sections that do not displace during static loading. The section or sections that do not displace against itself or themselves during the tensile-compression load or during the static torsion load are also determined with the help of this method.

***Keywords**: displacement, deformation, unsupported bars, cross sections*

1. Introduction

The cross sections of the straight bars subjected to axial forces displace, and the ones subjected to axial moments rotate in relation to their initial position prior to the loading. For the straight bars, supported in different points and axially loaded with forces or moments, the final positions of the cross sections can be determined if we take as a benchmark a section situated in one of these supports [1]. For the corresponding support, we know the fact that the displacement in that point is null. The finite element analysis software operates in the same way, and the conditions for contour displacement are boundary conditions on the basis of which the system of equations is verified. Normally, for unsupported bars subjected to axial loads with forces or moments, we consider one of the ends as being fixed and we draw a conventional diagram of the displacements, meant to establish the relative displacements between various sections of the bar. Thus, we can calculate the deformations in an error free modality, and then the stresses that are established in different points of the bar following loads [2]. On the basis of the method presented in this paper, we will be able to determine the *real* displacements (rotations) of the cross sections of the unsupported straight bars, subjected to axial loads (with forces or moments). For example, the determination of the real value for the rotation of cross sections is necessary for unsupported shafts, subjected to moments of torsion and on which various elements of transmission and/or of coupling are mounted [3].

2. Calculation ratios for the displacements of cross sections

The calculation of the displacements of the cross sections of straight bars is done by taking into consideration certain hypotheses [4]:

- the material these bars are made of is considered homogenous and it has isotropic behavior when subjected to loading;
- the applied load introduces deformations only in the elastic field;
- we consider that Bernoulli's hypothesis, according to which a plane and perpendicular section on the axis of the bar before bending will remain plane and perpendicular on the axis of the bar after the bending as well, is valid.

Let us consider a straight bar, with a constant section, fixed at one end and loaded at the other end with a force or moment sensed according to the geometrical axis of the bar, fig. 1.

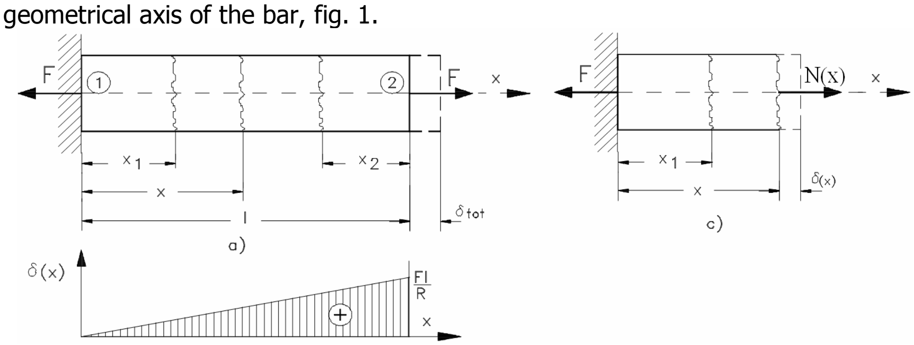

Figure 1. Straight bar subjected to axial load

Under the action of the load N, the bar bends, and, consequently, the various cross sections displace against the initial position. Thus, if the axial load is a force, the cross sections of the bar will displace and the bar will elongate by the quantity δ_{tot}, fig. 1.a. If the axial load is a moment, the cross sections of the bar will rotate against their initial position, and the end of the bar will rotate by the quantity $\delta_{tot.}$

From hereon, we will use the same notation, respectively δ, and the same notion, respectively displacement, for the displacements or rotations of the cross sections. The bar in figure 1a is made up of a single region (zone) on which we have the same variation law of sectional stresses N(x), fig. 1c. Hence, we know that the total displacement of the bar subjected to tensile or torsion stress will be given by the relation [5]:

$$\delta_{tot} = \int_0^l \frac{N(x)}{R(x)} dx \tag{1}$$

where N(x)=F represents the stress of the section x € [0, l] whereas R(x) represents the rigidity to the respective stress and is given by:
- for tensile-compression stress:

$R(x) = E \cdot A(x)$;
- for torsion stress

$R(x) = G \cdot I_p(x)$;

where:
- E and G represent the longitudinal, respectively the cross coefficient of elasticity;
- A(x) is the area of the cross section of the bar, noted with x;
- $I_p(x)$ represents the polar moment of inertia of the same section.

The displacement of a cross section situated at the distance x from the fixed end of the bar, fig. 1c, will be given by the relation:

$$\delta(x) = \int_0^x \frac{N(x_1)}{R(x_1)} dx_1 \tag{2}$$

or calculated in relation to the free end

$$\delta(x) = \delta_{tot} - \int_0^{(l-x)} \frac{N(x_2)}{R(x_2)} dx_2 \tag{3}$$

with x, x_1 and x_2 as in figure 1a.

Thus, the diagram of the displacements of the cross sections for the straight bar in figure 1a, subjected to tensile-compression stress or to torsion stress, resembles the one presented in figure 1b.

Various regions of the bar change when, along its length, either the variation law of the bar section or the variation law of the stress or both change. Hence, the displacement of a section of the bar will be given by the relation:

$$\delta(x_i) = \delta_i + \int_0^{x_i} \frac{N(x_{1i})}{R(x_{1i})} dx_{1i} \tag{4}$$

where x_i and x_{1i} are measured starting from the section i, δ_i representing the displacement of the same section.

2. The conventional diagrams of displacements

As we have already shown, in order to draw the conventional diagrams of the displacements δ of the cross sections for the straight bars subjected to tensile-compression stress or to torsion stress, it is necessary to know the displacement of a previous section. For the bars fixed in a specific section, we consider the displacement of that section (δ=0) as the reference. Hence, as shown in fig. 1b, we can draw the diagram of the displacements of the cross sections on the whole bar. For the free bars, where the displacement is not prevented in any of the sections, we cannot draw the diagram of the displacements based on the procedure described above because there are no initially known displacements of any of the sections. Nevertheless, in these cases we can calculate the relative displacements of the sections by arbitrarily considering one end of the bar as being fixed (δ_1=0), and by reporting all the other displacements in relation to this fixed end.

Let us consider the straight bar in fig. 2a, subjected to tensile-compression stress or to torsion stress, for which none of the sections is prevented from displacing. Consequently, we cannot know from the very beginning the displacement of any of the cross sections. Thus, we cannot draw a real diagram of the displacements, representing the displacements of the cross sections either against their initial position or against a bar section that does not displace during the stress from the zero load to the maximum load.

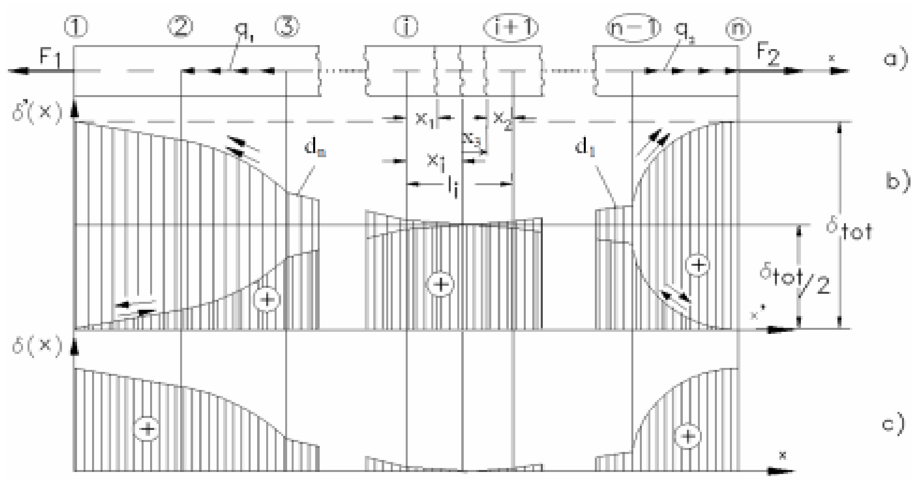

Figure 2. Axial load displacement diagram

For the bar in figure 2a we can draw two "conventional" diagrams of the displacements, considering first of all the end 1 as being fixed (δ_1=0) and then the end n as being fixed (δ_n=0). In figure 2b these diagrams are noted with $d_{1,}$ respectively d_n.

We may easily notice that the two conventional diagrams will intersect in at least one point. This statement is true for the following reasons:

- for each of the two diagrams, the starting point is "0", taken from the left, respectively from the right end of the bar;
- when the end 1 is fixed, the end n will displace by the quantity $\delta_n = \Delta l_{tot}$, where Δl_{tot} is the total elongation of the bar;
- when the end n is fixed, the end 1 will displace by the quantity $\delta_1 = \Delta l_{tot}$. Thus, we will have: $\delta_1 = \delta_n = \Delta l_{tot}$.

Consequently, if the trajectory followed by the two conventional diagrams is situated between the same values, 0 and Δl_{tot}, one starting from the right, and the other from the left, the two diagrams will definitely intersect, at least in one point.

Let us suppose that one of the intersection points of the two diagrams is situated within the region (i, i+1) of the bar, at the distance xi from the section i, figure 1. If we consider that the end 1 is fixed, and respectively that the end n is fixed, the displacement of the section x_i , in agreement with relation (4), will be in the two cases:

$$\begin{cases} \delta^1(x_i) = 0 + \delta_{12} + \ldots + \delta_{i-1,i} + \int_0^{x_i} \frac{N(x_1)}{R(x_1)} dx_1 \\ \delta^n(x_i) = 0 + \delta_{n,n-1} + \ldots + \delta_{i+2,i+1} + \int_0^{(l-x_i)} \frac{N(x_2)}{R(x_2)} dx_2 \end{cases} \tag{5}$$

where $\delta i_{-1,i}$ represents the relative displacement between the sections i-1 and i, being equal to the elongation (rotation) of the bar section (i-1,i).

If we suppose that the two conventional diagrams intersect in section $x_{i,}$ we will have:

$$\delta^1(x_i) = \delta^n(x_i) \tag{6}$$

and consequently:

$$\delta_{12} + \ldots + \delta_{i-1,i} + \int_0^{x_i} \frac{N(x_1)}{R(x_1)} dx_1 = \delta_{n,n-1} + \ldots + \delta_{i+2,i+1} + \int_0^{(l-x_i)} \frac{N(x_2)}{R(x_2)} dx_2 \tag{7}$$

4. Calculation of the displacements of the ends of the freely deformable bar

In the real case, where none of the sections of the bar is prevented from displacing following the deformation of the bar, its ends will displace by the quantities δ_1 and δ_n. Consequently, given the relation (3), the real displacement of the same section x_i, calculated from the left or from the right end of the bar, will be written under the form:

$$\begin{aligned} \delta^{REAL}(x_i) &= \delta_1 - \left(\delta_{12} + \ldots + \delta_{i-1,i} + \int_0^{x_i} \frac{N(x_1)}{R(x_1)} dx_1 \right) = \\ &= \delta_n - \left(\delta_{n,n-1} + \ldots + \delta_{i+2,i+1} + \int_0^{(l-x_i)} \frac{N(x_2)}{R(x_2)} dx_2 \right) \end{aligned} \tag{8}$$

In relation (8), the first equality is written when we consider the end 1 as being fixed, whereas, for the second equality, the n end is considered to be fixed.

According to relation (7) we consider that the terms that appear between brackets in the relation (8) are equal. Hence, for the straight bar which suffers a free deformation when subjected to static axial loading, we can write the relation:

$$\delta_1 = \delta_n \tag{9}$$

From the relation (9) we can draw a very important conclusion, namely that, for a bar which freely deforms in all sections, the displacements of the ends of this bar are equal. We already know that, as concerns the bar with a fixed section, as well as the freely deformable bar, the relation below is valid:

$$\delta_1+\delta_n=\delta_{1n}=\delta_{tot} \tag{10}$$

where:

- δ_1 and δ_n represent the displacements of the free ends of the unsupported bar;
- δ_{1n} represents the displacement of one end of the bar when the other end is fixed;
- δ_{tot} represents the total elongation of the bar when one of the ends is fixed.

Given the relations (9) and (10), we will have

$$\delta_1 = \delta_n = \frac{\delta_{tot}}{2} \tag{11}$$

where the total elongation of the bar, δ_{tot}, can be determined by means of calculation or by drawing a conventional diagram.

Thus, the real displacement of the section x_i on the freely deformable bar, the section at the intersection of the conventional diagrams, will be given by the relation:

$$\begin{aligned}
\delta^{\text{Real}}(x_i) &= \frac{\delta_{tot}}{2} - \left(\delta_{12} + \ldots + \delta_{i-1,i} + \int_0^{x_i} \frac{N(x_1)}{R(x_1)} dx_1\right) = \\
&= \frac{1}{2}\left(\delta_{12} + \ldots + \delta_{i-1,i} + \int_0^{x_i} \frac{N(x_1)}{R(x_1)} dx_1 + \int_{x_i}^{l_i} \frac{N(x_3)}{R(x_3)} dx_3 + \delta_{i+1,i+2} + \ldots + \delta_{n,n-1}\right) - \\
&- \left(\delta_{12} + \ldots + \delta_{i-1,i} + \int_0^{x_i} \frac{N(x_1)}{R(x_1)} dx_1\right) = \\
&= \frac{1}{2}\left(\delta_{n,n-1} + \ldots + \delta_{i+1,i+2} + \int_{x_i}^{l_i} \frac{N(x_3)}{R(x_3)} dx_3\right) - \frac{1}{2}\left(\delta_{12} + \ldots + \delta_{i-1,i} + \int_0^{x_i} \frac{N(x_1)}{R(x_1)} dx_1\right)
\end{aligned} \tag{12}$$

where the equality:

$$\int_{x_i}^{l_i} \frac{N(x_3)}{R(x_3)} dx_3 = \int_0^{(l_i - x_i)} \frac{N(x_2)}{R(x_2)} dx_2 \tag{13}$$

is obvious because of the fact that each of the terms represents the deformation of the same bar segment, comprised between sections x_i and l_i, fig. 2a.

If, in relation (12) we consider the equalities given by the relations (13) and (7), we will obtain:

$$\delta^{\text{Real}}(x_i) = 0 \tag{14}$$

Therefore, when none of the cross sections of a straight bar is prevented from displacing under the action of static axial loading, there will be at least one section that will not displace against its initial position. This section is situated next to the intersection of the "conventional" diagrams of the displacement that are drawn by considering, in turn, the ends of the bar as being fixed. On the other hand, the relations (9), (5) and (11) lead to the following equalities:

$$\delta_1^{\mathrm{Real}} = \delta^1(x_i) = \delta^n(x_i) = \delta_{\mathrm{n}}^{\mathrm{Real}} = \frac{\delta_{tot}}{2} \tag{15}$$

where:

- δ_1^{Real} and $\delta_{\mathrm{n}}^{\mathrm{Real}}$ are the real displacements of the ends of the freely deformable bar;
- $\delta^1(x_i)$ and $\delta^n(x_i)$ represent the displacement of the section x_i within the "conventional" diagram;
- δ_{tot} represent the total elongation of a bar, which can be determined on the basis of a conventional diagram.

From the relation (15) we can draw the following conclusions:

- the real displacements of the ends of the freely deformable bar, subjected to axial loading, are equal between them, as well as equal to the displacement of the section x_i that does not displace ($\delta_{(x_i)}^{\mathrm{Real}} = 0$) against its initial position.
- the same displacements of the ends of the bar are equal to half of the total displacement (δ_{tot}) which can be determined on the basis of a conventional diagram.

5. The real diagram of displacements

We will continue by presenting the work modality for the drawing of the real diagram of the displacements of cross sections for a straight bar, without deformation restrictions, subjected to tensile-compression stress or to torsion stress, figure 3a.

Given the already discussed aspects, the drawing of the real diagram of the displacements involves the following steps:

1 – we trace a single "conventional" diagram of the displacements, considering, for example, the left end as the fixed one;

2 – we trace a new abscissa in relation to the first one, at the distance $\frac{\delta_{tot}}{2}$, fig. 3b;

3 – the intersection point or points of the new abscissa with the conventional diagram will be situated next to the section of the bar that does not displace against its initial position;

4 – a movement direction is marked on the conventional diagram, from zero to δ_{tot};

5 – on the same conventional diagram we mark a new movement direction, this time starting from one of the sections that do not displace (intersection with the new abscissa) towards the ends of the bar;

6 – when the direction marked at point 4 does not coincide to the direction marked at point 5, the conventional diagram must be reversed on the other side of the new abscissa, figure 3b.

Thus, we obtain the real diagram of the displacements, which, in figure 3b, is represented above the new abscissa x.

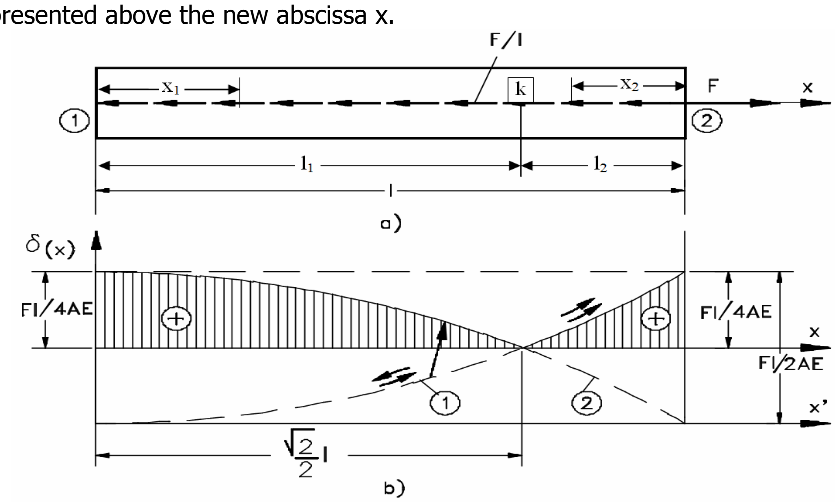

Figure 3. Example concerning the drawing modality of the real diagram for the displacements of an axially loaded unsupported straight bar

From this example we may notice that the section that does not displace against its initial position when the axial load increases from zero to the maximum load is situated at the distance $\frac{\sqrt{2}}{2}l$ in relation to the left end of the bar. We may also notice that the new abscissa x intersects the conventional diagrams at the distance $\frac{\delta_{tot}}{2}$ where $\delta_{tot} = \frac{Fl}{2AE}$, **A** being the area of the cross section of the bar. The displacements of the free ends of the bars have the same values, respectively $\delta_1 = \delta_2 = \frac{Fl}{4AE}$.

As we may notice, in order to draw the final diagram of the displacements, if we have the conventional diagram, we must act in the following manner:

- we draw the movement direction of the conventional diagram from point 1 (considered fixed) to point 2 (considered free).
- we draw another movement direction of the same diagram, starting from the intersection with the new abscissa x.
- where the two directions do not coincide, the diagram is reversed in relation to abscissa x.

6. Analytical calculation of the position of the non-displaced section.

In this paragraph we will analytically determine the position of the section that does not displace against its initial position during the application of the load from zero to the maximum value. For comparison, this calculation will be performed for the example in figure 3a. As we have previously noticed, the displacements of the free ends of the bars are equal, $\delta_1=\delta_2$. Consequently, if the section situated at the distance l_1 from the left end does not displace, we obtain the equality $\Delta l_1=\Delta l_2$ in which Δl_1 and Δl_2 represent the elongations of the sections 1-k and 2-k. These elongations can be calculated with the help of the known relations.

$$\Delta l_1 = \int_0^{l_1} \frac{N(x_1)}{R(x_1)} dx_1 \tag{16}$$

$$\Delta l_2 = \int_0^{l_2} \frac{N(x_2)}{R(x_2)} dx_2 \tag{17}$$

where $N(x_1)$ and $N(x_2)$ are the sectional stresses on the sections 1-k and 2-k. These can be written under the form:

$$N(x_1) = \frac{F}{l} x_1 \tag{18}$$

$$N(x_2) = F - \frac{F}{l} x_2 \tag{19}$$

Thus, for the elongations of the bar sections 1-k and 2-k, we will have the relations:

$$\Delta l_1 = \int_0^{l_1} \frac{F x_1}{R(x_1) \cdot l} dx_1 = \frac{F \cdot l_1^2}{2R \cdot l} \tag{20}$$

$$\Delta l_2 = \int_0^{l_2} \frac{F - \frac{F \cdot x_2}{l}}{R(x_2)} dx_2 = \frac{F}{2R \cdot l}\left(2l \cdot l_2 - l_2^2\right) \tag{21}$$

where $R(x_1)= R(x_2)= R=$const. for the analyzed case.

By putting the equal sign between the relations (20) and (21) we will obtain:

$$l_1^2 = 2l \cdot l_2 - l_2^2 \tag{22}$$

Given the obvious relation $l1+l_2=l$, from relation (22) we obtain:

$$2l_1^2 - l^2 = 0 \tag{23}$$

And thus we get:

$$l_1 = \frac{\sqrt{2}}{2} l \tag{24}$$

value which corresponds to the value that was graphically determined in the previous paragraph.

4. Conclusions

Given the fact that, most of the times, for axially loaded bars, we are interested in deformations and strains, for a freely deformable bar we can draw a conventional diagram by arbitrarily considering one of the ends as being fixed and by thus determining the relative displacements between two sections.

There are, nevertheless, practical cases in which we have to determine the real position of the cross sections for a straight bar subjected to static axial loads when the bar is not supported in any points. This paper describes a method for the determination of this position. On the basis of this method we also determine the section or the sections that do not displace during the static axial loading, with the load increasing from zero to the maximum value. The method is a simpler one, relying on the drawing of a "conventional" diagram by considering one of the ends of the bar as being fixed. Then, a new abscissa is drawn at $\delta_{tot}/2$ in relation to the first abscissa, the intersection point or points for the new abscissa with the conventional diagram representing the position of the section or of the sections that do not displace against their initial position. Moreover, we must perform a swinging of one portion of the "conventional" diagram in relation to the new abscissa, as shown in the previous paragraphs. Thus we obtain the real diagram of the displacements of the cross sections, as well as the position of the sections that do not displace during the loading. It is obvious that, as concerns the freely deformable straight bars subjected to axial loads, these sections do exist and are emphasized with the help of this method.

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