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The Computation of the Efforts into the Plan System of the Elastic Bars which Suspends a Rigid with Constraints

One considers that the rigid with constraints acted by the external forces suported by the elatic bars embedded at the base. Under the influence of the external forces appear the connection forces and the displacements in the kinematic pairs. One deduces the mathematical model for the computation of the reactions and the displacements and one makes a numerical applications.

Keywords: *rigid, constraints, elastic, joint bars*

1. Introduction

In the paper [a], using the plückerian co-ordinates and the relative displacements method [8], it elaborated the computation model of the efforts from elastic bars spatially arranged, bars that suspended a rigid with constraints without clearances.

In this paper it will customize the general model for the plane elastic bars system and it will be made an application for such a system.

2. Notations. General computation relations

One considers that the rigid body relative to the axis system O_0XYZ (fig. 1), suspended by the elastic bars A_iB_i embedded in thepoints A_i, B_i and one considers that the rigid has constraints without clearances.

An example of such constraint is made (fig. 1), when the point D of the rigid remains located on the fixed rigid surface equation $g(X,Y,Y)=0$

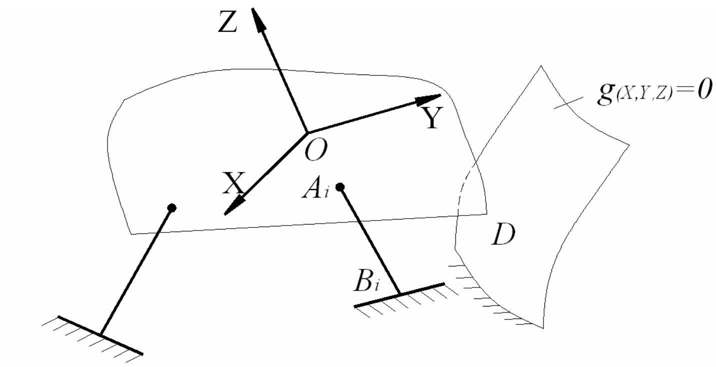

Figure 1.

Using the general notations [1]

- $[K_i]$ the stiffness matrix of the bar A_iB_i relative to the reference system $OXYZ$
- $[K]$ the general stiffness matrix of the elastic bars system

$$[k]=\sum_{i=1}^{n}[k_i], \tag{1}$$

- $[S]$ the possible displacements matrix from constraints
- $[\xi]$ the column matrix of the scalar parameters of the displacements from constraints
- $[U]$ the restrictions matrix from constraints
- $[\eta]$ the column matrix of the scalar parameters of the restrictions from constraints
- $[F]$ the column matrix of the components of the external forces torsor which act the rigid

one obtains [1] the matrix equation of the elastic equilibrium

$$\{F\}+[U]\{\eta\}=[K][S]\{\xi\} \tag{2}$$

and taking into account the conditions

$$[U]^T[S]=[0]\,;\;[S]^T[U]=[0] \tag{3}$$

from (2) one deduces the matrices of the scalar parameters

$$\{\xi\}=\left[[S]^T\cdot[K]\cdot[S]\right]^{-1}\cdot[S]^T\cdot\{F\} \tag{4}$$

$$\eta=-\left[[U]^T\cdot[K]^{-1}[U]\right]^{-1}[U]^T[k]^{-1}\{F\} \tag{5}$$

The efforts $\{F_i\}$ from bars relative to the local reference systems $A_ix_iy_iz_i$ are given by the equalities

$$\{F_i\} = [T_i]^{-1}[k_i][S][\xi], \; i = 1,2,.....n \tag{6}$$

3. The plan bars system

3.1. ***The stiffness matrix***

In the case of the plan bars system (fig. 2)

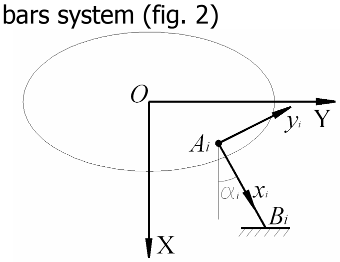

Figure 2.

using the notations

- $A_i y_i, A_i z_i$ the inertial central principal axes of the normal section in A_i by the bar $A_i B_i$
- $A_i x_i$, the axis of the centres of gravity of the normal sections, sections considered identical
- l_i, the lenght of the bar $A_i B_i$
- $\tilde{A}_i$, the area of the normal section
- I_i, the inertial moment of the normal section relative to the axis $A_i z_i$
- E_i, the Young modulus
- k_{1i}, k_{2i}, the stiffness defined by the relations

$$k_{1i} = \frac{E_i A_i}{l_i} \; ; \; k_{2i} = \frac{E_i I_i}{l_i^3} \tag{7}$$

- $[k_i]$, the local stiffness matrix of the bar $A_i B_i$

$$[k_i] = \begin{bmatrix} k_{1i} & 0 & 0 \\ 0 & 12k_{2i} & 6k_{2i}l_i \\ 0 & 6k_{2i}l_i & 4k_{2i}l_i^2 \end{bmatrix} \tag{8}$$

- α_i, the angle between axes OX, $A_i B_i$

- (X_i, Y_i), the co-ordinates of the point A_i
- $[T_i]$, $[\tilde{T}_i]$, the position matrices

$$[T_i]=\begin{bmatrix} \cos\alpha_i & -\sin\alpha_i & 0 \\ \sin\alpha_i & \cos\alpha_i & 0 \\ X_i\sin\alpha_i - Y_i\cos\alpha_i & X_i\cos\alpha_i + Y_i\sin\alpha_i & 1 \end{bmatrix},$$

$$[\tilde{T}_i]=\begin{bmatrix} \cos\alpha_i & -\sin\alpha_i & Y_i \\ \sin\alpha_i & \cos\alpha_i & X_i \\ 0 & 0 & 1 \end{bmatrix} \quad (9)$$

- $[K_i]$, the stiffness matrix of the bar A_iB_i in the OXY system

$$[K_i]=[T_i][k_i][\tilde{T}_i]^{-1} \quad (10)$$

where

$$[\tilde{T}_i]^{-1}=[T_i]^T \ ; \ [T_i]^{-1}=[\tilde{T}_i]^T \quad (11)$$

one obtains the stiffness matrix of the elastic bars system

$$[K]=\sum_{i=1}^{n}[K_i] \quad (12)$$

3.2. The restrictions matrices

For the restrictions matrix computation one takes into account the relation

$$[U]=[T][u] \quad (13)$$

where:

- $[u]$, is the restrictions matrix in the loacal reference system
- $[U]$, is the restrictions matrix in the general reference system OXY
- $[T]$, the position forces matrix of the local system relative to general reference system.

If the local reference system has the axes parallel to the axes OX, OY and the origin has the co-ordinates (X, Y), the position matrix is

$$[T]=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -Y & X & 1 \end{bmatrix} \quad (14)$$

For the bearing constraint of the rigid on the fixed rigid equation curve $g(X,Y)=0$ in point $D(X_D,Y_D)$, denoting with g_X, g_y the partial derivatives calculated in the point D, there are obtained the expressions

$$\{u\}=\begin{bmatrix} g_X^* \\ g_Y^* \\ 0 \end{bmatrix};\ \{U\}=\begin{bmatrix} g_X^* \\ g_Y^* \\ X_D g_Y^* - Y_D g_X^* \end{bmatrix}; \tag{15}$$

where

$$g_X^* = \frac{g_X}{\sqrt{g_X^2+g_Y^2}}\ ;\ g_Y^* = \frac{g_Y}{\sqrt{g_X^2+g_X^2}} \tag{16}$$

and for the fixed point constraint (joint) in $D(X_D,Y_D)$, there are obtained the expressions

$$[u]=\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix};\ [U]=\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -Y_D & X_D \end{bmatrix} \tag{17}$$

3.3. ***The displacements matrix***

For the matrix (small) displacements computation one takes into account the relation

$$[S]=[\tilde{T}][s] \tag{18}$$

where:

- $[s]$, is the displacements matrix in the local reference system
- $[S]$, is the displacements matrix in the general reference system OXY
- $[\tilde{T}]$, is the position matrix for displacements of the local system relative to the system OXY

If the local reference system has its axes parallel to OX, OY, the origin has the co-ordinates (X,Y) the position matrix is

$$[\tilde{T}]=\begin{bmatrix} 1 & 0 & Y \\ 0 & 1 & -X \\ 0 & 0 & 1 \end{bmatrix} \tag{19}$$

For the bearing constraint of the rigid on the fixed rigid curve equation $g(X,Y)=0$ in point $D(X_D,Y_D)$,there are obtained the expressions

$$\{s\}=\begin{bmatrix}-g_Y^* & 0\\ g_X^* & 0\\ 0 & 1\end{bmatrix};\ \{S\}=\begin{bmatrix}-g_Y^* & Y_D\\ g_X^* & -X_D\\ 0 & 1\end{bmatrix}; \tag{20}$$

and for the fixed point constraint (joint) in $D(X_D,Y_D)$ there are obtained the expressions

$$\{s\}=\begin{bmatrix}0\\0\\1\end{bmatrix};\ \{S\}=\begin{bmatrix}Y_D\\ -X_D\\ 1\end{bmatrix} \tag{21}$$

4. Application

One considers a rigid straight bar A_1A_2 from fig. 3, of lenght $2b$ propped in A_1 on the rigid equation straight line $X\cos\beta - Y\sin\beta + b\sin\beta = 0$

The rigid bar A_1A_2 embedded bars in $B_i, i=1{,}2$ at the base and in $A_i, i=1{,}2$ at the rigid bar.

If the force F acts upon the rigid bar on the axis direction OY let us determinate the displacements ξ_1,ξ_2 the reaction N from the point A_1 and the efforts from bars.

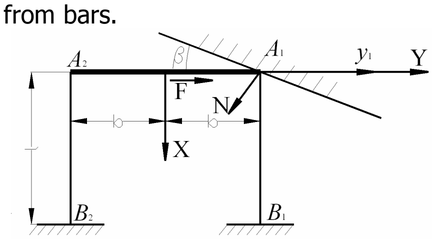

Figure 3.

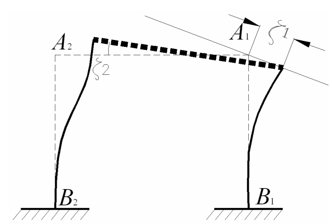

Figure 4.

Choosing the reference system from fig.3 it results the equalities

$$k_{11}=k_{12}=k_1=\frac{EA}{l};\ k_{21}=k_{22}=k_2=\frac{EI}{l^3};\ \alpha_1=\alpha_2=0;$$
$$g_X^*=\cos\beta;\ g_Y^*=-\sin\beta \tag{22}$$

$$[k_i]=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ (-1)^i b & 0 & 1\end{bmatrix}\begin{bmatrix}k_1 & 0 & 0\\ 0 & 12k_2 & 6k_2 l\\ 0 & 6k_2 l & 4kl^2\end{bmatrix}\begin{bmatrix}1 & 0 & (-1)^i b\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \tag{23}$$

$$[K]=[K_1]+[K_2]=2\cdot\begin{bmatrix}k_1 & 0 & 0\\ 0 & 12k_2 & 6k_2 l\\ 0 & 6k_2 l & 4kl^2+k_1b^2\end{bmatrix} \tag{24}$$

$$\{U\}=\begin{bmatrix}\cos\beta\\ -\sin\beta\\ -y\cos\beta\end{bmatrix};\ [S]=\begin{bmatrix}\sin\beta & b\\ \cos\beta & 0\\ 0 & 1\end{bmatrix} \tag{25}$$

$$D=k_1^2b^2\sin^2\beta+12k_2^2l^2\cos^2\beta+ \\ 4k_1k_2\left(l^2\sin^2\beta+6b^2\cos^2\beta-3bl\sin\beta\cos\beta\right) \tag{26}$$

$$\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix}=\frac{F\cos\beta}{2D}\begin{bmatrix}4k_2l^2+2k_1b^2\\ -(k_1b\sin\beta+6k_2l\cos\beta)\end{bmatrix} \tag{27}$$

$$N=\eta=\frac{Fk_1}{D}\left[\left(4k_2l^2+k_1b^2\right)\sin\beta-6k_2lb\cos\beta\right] \tag{28}$$

$$\{F_1\}=\begin{bmatrix}k_1(\xi_1\sin\beta+\xi_2\cos\beta)\\ 12k_2\xi_1\cos\beta+6k_2l\xi_2\\ 6k_2l\xi_1\cos\beta+\left(4k_2l^2+k_1b^2\right)\xi_2\end{bmatrix};$$

$$\{F_2\}=\begin{bmatrix}k_1(\xi_1\sin\beta-\xi_2 b)\\ 12k_2\xi_1\cos\beta+6k_2l\xi_2\\ 6k_2l\xi_1\cos\beta+\left(4k_2l^2+k_1b^2\right)\xi_2\end{bmatrix} \tag{29}$$

$$tg\beta_0=\frac{6k_2lb}{4k_2l^2+k_1b^2} \tag{30}$$

If $0<\beta<\beta_0$ the reaction N is negative, if $\beta=\beta_0$ the reaction N is null and if $\beta>\beta_0$ the reaction N is positive.

In a numerical application if one considers that the elastic bars with $E=2\cdot10^{11}\mathrm{N}\cdot\mathrm{m}^{-2}; l=0.8\mathrm{m}$, have a circular section $d=16\mathrm{mm}$, that $\beta=10^\circ$, $F=5000\mathrm{N}$ and the rigid bar is $b=0.4\mathrm{m}$, there are obtained the results:

$\beta_0=0.5\cdot10^{-4}\mathrm{rad},\ N=28802\,\mathrm{N},\ \xi_1=3.27\ \mathrm{rad},\ \xi_2=7.09\ \mathrm{rad}$

$$\{F_1\} = 5 \cdot 10^3 \cdot \begin{bmatrix} 3.4\,\mathrm{N} \\ 0 \\ 2.836\ \mathrm{rad} \end{bmatrix};\ \{F_2\} = 5 \cdot 10^3 \cdot \begin{bmatrix} -2.26\,\mathrm{N} \\ 0 \\ 2.83\ \mathrm{rad} \end{bmatrix}$$

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