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The Equilibrium of the Solid Rigid with Constraints

In our paper we obtain the matrical equations of equilibrium for a solid rigid with constraints. These equations are deduced from the Lagrange equations. Finally we present a completely solved example.

***Keywords**: Lagrange equations, pseudo-coordinates*

The Lagrange equations

Let us consider a rigid for which the rotational matrix is

$$\mathbf{A} = [\boldsymbol{\psi}][\boldsymbol{\theta}][\varphi], \tag{1}$$

where ψ, θ, φ are the rotational angles.

Denoting by $\{\mathbf{q}\}$ the column matrix of the generalized co-ordinates and considering that the rigid has p holonomic constraint,

$$f_i(\mathbf{q}) = f_i(X_O, Y_O, Z_O, \psi, \theta, \varphi),\ i = \overline{1, p}. \tag{2}$$

where X_O, Y_O, Z_O are the co-ordinates of the origin O of the mobile system linked to the origin $Oxyz$ relative to a fixed system O_0XYZ, denoting by

$$[\mathbf{B}] = \frac{D(f_1, \dots, f_6)}{D(q_1, \dots, q_6)} = \begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \dots & \frac{\partial f_6}{\partial q_6} \\ \dots & \dots & \dots \\ \frac{\partial f_p}{\partial q_1} & \dots & \frac{\partial f_p}{\partial q_6} \end{bmatrix}, \tag{3}$$

the matrix of constraints, we obtain the equality

$$[\mathbf{B}]\{\dot{\mathbf{q}}\} = \{\mathbf{0}\}, \tag{4}$$

or, in more general form,

$$[\mathbf{B}]\{\dot{\mathbf{q}}\} = \{\mathbf{C}\}, \tag{5}$$

where $\{\mathbf{C}\}$ is a column matrix that depends on the generalized co-ordinates q_i, $i = \overline{1, 6}$.

If T is the kinetic energy, writing

$$\mathbf{B} = \begin{bmatrix} B_{11} & B_{12} & \dots & B_{1n} \\ \dots & \dots & \dots & \dots \\ B_{p1} & B_{p2} & \dots & B_{pn} \end{bmatrix}, \tag{6}$$

where n is the number of the generalized co-ordinates, then the Lagrange equations read

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} + F_{q_k} + \sum_{i=1}^{p} B_{ik}\lambda_i \ , \ k = \overline{1,n}, \tag{7}$$

where λ_1, ..., λ_p are the Lagrange multipliers.

The equations (7) can be put in a matrical form as follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\mathbf{q}}}\right) - \left\{\frac{\partial T}{\partial \dot{\mathbf{q}}}\right\} = \{\mathbf{F_q}\} + [\mathbf{B}]^T\{\boldsymbol{\lambda}\}, \tag{8}$$

with

$$\{\mathbf{q}\} = [q_1 \ \dots \ q_n]^T, \ \{\dot{\mathbf{q}}\} = [\dot{q}_1 \ \dots \ \dot{q}_n]^T, \tag{9}$$

$$\{\boldsymbol{\lambda}\} = [\lambda_1 \ \dots \ \lambda_p]^T, \tag{10}$$

$$\{\mathbf{F_q}\} = [F_{q_1} \ F_{q_2} \ \dots \ F_{q_n}]^T, \tag{11}$$

F_{q_i} being the generalized force corresponding to the co-ordinates q_i, $i = \overline{1,n}$.

The equilibrium equations for a solid rigid with constraints

In the case of mechanical systems with holonomic constraints, the kinetic energy at equilibrium is zero, from Lagrange equations one gets the matrical form

$$\{\mathbf{F_q}\} + [\mathbf{B}]^T\{\boldsymbol{\lambda}\} = \{\mathbf{0}\}. \tag{12}$$

At this equation we add the matrical constraining function

$$\{\mathbf{f}\} = \begin{bmatrix} f_1(q_1,\dots,q_n) \\ \dots \\ f_1(q_1,\dots,q_n) \end{bmatrix} = \begin{bmatrix} 0 \\ \dots \\ 0 \end{bmatrix}. \tag{13}$$

For a solid rigid we obtain the same equations with

$$\{\mathbf{q}\} = [X_O \ Y_O \ Z_O \ \psi \ \theta \ \varphi]^T. \tag{14}$$

The case of pseudo-coordinate

We consider the case when the p constraining functions depend on the basic generalized co-ordinates q_1, ..., q_n and on the generalized pseudo-coordinates ξ_1, ..., ξ_k, that is

$$f_i(q_1,...,q_n,\xi_1,...,\xi_k) = 0\ ,\ i = \overline{1,p}\ . \tag{15}$$

Deriving with respect to time, denoting by $[\mathbf{B}]$ the Jacobi matrix, using the notations

$$\{\mathbf{q}\} = [X_O\ \ Y_O\ \ Z_O\ \ \psi\ \ \theta\ \ \varphi]^T\ ,\ \{\xi\} = [\xi_1\ \ \xi_2\ \ ...\ \ \xi_k]^T \tag{16}$$

$$[\mathbf{B}] = \begin{bmatrix} [\mathbf{B}_{11}] & [\mathbf{B}_{12}] \\ [\mathbf{B}_{21}] & [\mathbf{B}_{22}] \end{bmatrix}, \tag{17}$$

where $[\mathbf{B}_{11}]$ has $p-k$ lines and 6 columns, $[\mathbf{B}_{12}]$ has $p-k$ lines and k columns, $[\mathbf{B}_{21}]$ has k lines and 6 columns and $[\mathbf{B}_{22}]$ has k lines and k columns, one obtains the equality

$$\begin{bmatrix} [\mathbf{B}_{11}] & [\mathbf{B}_{12}] \\ [\mathbf{B}_{21}] & [\mathbf{B}_{22}] \end{bmatrix} \begin{Bmatrix} \{\dot{\mathbf{q}}\} \\ \{\dot{\xi}\} \end{Bmatrix} = \{\mathbf{0}\}, \tag{18}$$

which separates in the equations

$$\begin{aligned} &[\mathbf{B}_{11}]\{\dot{\mathbf{q}}\} + [\mathbf{B}_{12}]\{\dot{\xi}\} = \{\mathbf{0}\}, \\ &[\mathbf{B}_{21}]\{\dot{\mathbf{q}}\} + [\mathbf{B}_{22}]\{\dot{\xi}\} = \{\mathbf{0}\}. \end{aligned} \tag{19}$$

Considering that the matrix $[\mathbf{B}_{22}]$ is invertible, from (19) with the notation

$$[\mathbf{B}^*] = [\mathbf{B}_{11}] - [\mathbf{B}_{12}][\mathbf{B}_{22}]^T[\mathbf{B}_{21}], \tag{20}$$

one deduces the equality

$$[\mathbf{B}^*]\{\dot{\mathbf{q}}\} = \{\mathbf{0}\}, \tag{21}$$

and thus the pseudo-coordinates are eliminated and the matrical equation of equilibrium (12) becomes

$$\{\mathbf{F_q}\} + [\mathbf{B}^*]^T\{\lambda\} = \{\mathbf{0}\}, \tag{22}$$

where the matrix $\{\lambda\}$ has $p-k$ elements.

Application

Determine the equilibrium position for the elliptical shell drawn in figure 1, of semi-axes a, b and weight G, supported at O_0 on the axis O_0X and acted at the point A ($OA = d$)by another force P.

We consider the parametrical equations of the ellipse

$$x = a\cos\xi\ ,\ \ y = b\sin\xi\ . \tag{23}$$

The basic generalized co-ordinates are X_O, Y_O, φ, and ξ is a pseudo-coordinate.

The contact at O_0 leads to the equations

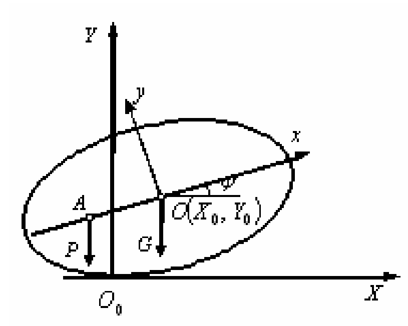


Figure 1. Elliptical plate.

$$X_O + a\cos\varphi\cos\xi - b\sin\varphi\sin\xi = 0\,,\ Y_O + a\sin\varphi\cos\xi + b\cos\varphi\cos\xi = 0\,, \qquad (24)$$

and the tangency at O_0 leads to the equation

$$-a\sin\varphi\sin\xi + b\cos\varphi\cos\xi = 0\,. \qquad (25)$$

One gets the matrices

$$[\mathbf{B}_{11}] = \begin{bmatrix} 1 & 0 & a\sin\varphi\cos\xi - b\cos\varphi\sin\xi \\ 0 & 1 & a\cos\varphi\cos\xi - b\cos\varphi\sin\xi \end{bmatrix}, \qquad (26)$$

$$[\mathbf{B}_{12}] = \begin{bmatrix} -a\cos\varphi\sin\xi - b\sin\varphi\cos\xi \\ -a\sin\varphi\sin\xi + b\cos\varphi\cos\xi \end{bmatrix}, \qquad (27)$$

$$[\mathbf{B}_{21}] = [0\ \ 0\ \ -a\cos\varphi\sin\xi - b\sin\varphi\cos\xi], \qquad (28)$$

$$[\mathbf{B}_{22}] = [-a\sin\varphi\cos\xi - b\cos\varphi\sin\xi] \qquad (29)$$

The potential energy is

$$V = GY_O + P(Y_O - d\sin\varphi) \qquad (30)$$

and leads us to the matrix of the generalized forces

$$\{\mathbf{F_q}\} = \begin{bmatrix} 0 \\ -G-P \\ Pd\cos\varphi \end{bmatrix}. \qquad (31)$$

We obtain the system

$$\begin{bmatrix} 0 \\ -G-P \\ Pd\cos\varphi \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -a\sin\varphi\cos\xi - b\cos\varphi\sin\xi & a\cos\varphi\cos\xi - b\sin\varphi\sin\xi \end{bmatrix}\cdot$$
$$\cdot\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -a\cos\varphi\sin\xi - b\sin\varphi\cos\xi \end{bmatrix}\lambda^* = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad (32a)$$

$$\left[a\cos\varphi\sin\xi - b\sin\varphi\cos\xi \;\; 0\right]\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix} + \left(-a\sin\varphi\cos\xi - b\cos\varphi\sin\xi\right)\lambda^* = 0, \qquad (32b)$$

$$X_O + a\cos\varphi\cos\xi - b\sin\varphi\sin\xi = 0, \qquad (32c)$$

$$Y_O + a\sin\varphi\cos\xi + b\cos\varphi\cos\xi = 0, \qquad (32d)$$

$$-a\sin\varphi\sin\xi + b\cos\varphi\cos\xi = 0. \qquad (32e)$$

First of all, one deduces the equalities

$$\lambda_1 = 0,\; \lambda_2 = G + P,\; \lambda^* = 0 \qquad (33)$$

$$Pd\cos\varphi + \left(a\cos\varphi\cos\xi - b\sin\varphi\sin\xi\right)\left(G + P\right) = 0 \qquad (34)$$

and it results the expression

$$X_O = \frac{Pd\cos\varphi}{P+G} = \frac{\left(a^2 - b^2\right)\cos\varphi\sin\varphi}{\sqrt{a^2\sin^2\varphi + b^2\cos^2\varphi}}, \qquad (35)$$

from which we get

$$\tan\varphi = \frac{Pdb}{\sqrt{(P+G)^2\left(a^2-b^2\right)^2 - P^2a^2d^2}},\; \cos\varphi = 0,\; P < \frac{G\left(a^2-b^2\right)}{da - \left(a^2-b^2\right)}, \qquad (36)$$

$$P > 0.$$

Further on, it follows

$$Y_O = \sqrt{a^2\sin^2\varphi + b^2\cos^2\varphi}, \qquad (37)$$

$$\operatorname{cotan}\xi = \frac{a}{b}\tan\varphi. \qquad (38)$$

If the ellipse becomes circle, $a = b = R$, then the previous formulas offer $X_O = 0$, $\cos\varphi = 0$, $\varphi = \frac{\pi}{2}$ or $\varphi = -\frac{\pi}{2}$, $Y_O = R$, $\xi = \pi$ or $\xi = 0$.

The two equilibrium positions are drawn in figure 2.

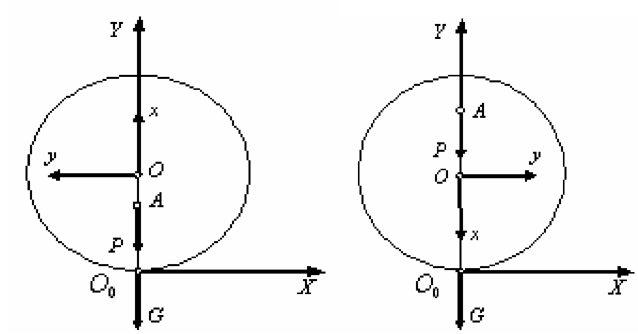
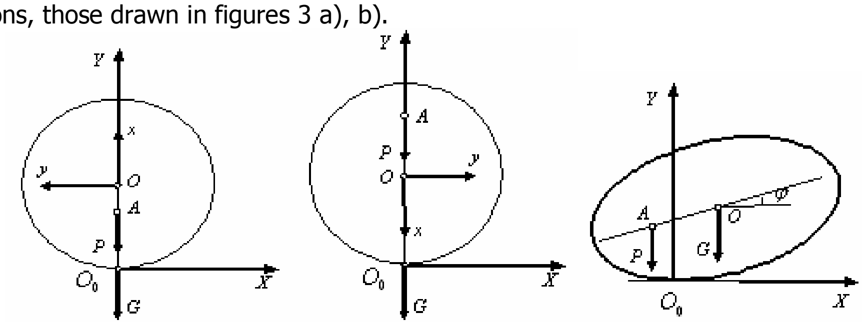


Figure 2. The equilibrium positions in case of the circle.

In the case of the ellipse there exist two or three equilibrium positions:

– if $P < \dfrac{G\left(a^2 - b^2\right)}{da - \left(a^2 - b^2\right)}$, $P > 0$, then there exist three equilibrium positions captured in figures 3 a), b), c);

– if $P \geq \dfrac{G\left(a^2 - b^2\right)}{da - \left(a^2 - b^2\right)}$, $P > 0$, then there exist only two equilibrium positions, those drawn in figures 3 a), b).



a) b) c)

Figure 3. The equilibrium positions in case of the ellipse.

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