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Two Degrees of Freedom Non-linear Model to Study the Automobile's Vibrations

In this paper we present a non-linear model for the study of an automobile's vibrations. The model has two degrees of freedom and it is highly non-linear. The forces in the springs are considered to be given by a polynomial potential. The equations of motion are obtained using the Lagrange second order equations. We determined the equilibrium positions. We proved the conditions for the uniqueness of the equilibrium. In our paper we studied the stability of the equilibrium and the stability of the motion. Finally a numerical application is presented.

***Keywords**: non-linear, vibrations, stability, automobile*

Introduction

There exist many models in the specific literature [1] to study the automobiles' vibrations. Their great majority are linear, the non-linear being almost absent. But there exist situation when the linear model are not appropiate, their results are not in accordance to the reality. The characteristics of the elastic elements can be considered linear only on small zones; when the elastic deformations are large, then the characteristics have to be considered non-linear, as they are in fact. The non-linear models lead to inconveniences to characterize the automobile4's behavior, and, generally speaking, their exact solutions can not be obtained.

Mathematical model

We shall consider an automobile schematized by the bar AB linked to the ground with the springs of non-linear characteristics A_0A and B_0B (fig. 1). The bar is homogenous of mass m, length $2L$, its centre of weight is situated at the middle of the distance AB, $AC = L$, $CB = L$, and the moment of inertia relative to a horizontal axis that passes through C is J. The springs A_0A and B_0B are identical and their length in non-deformed status is l_0.

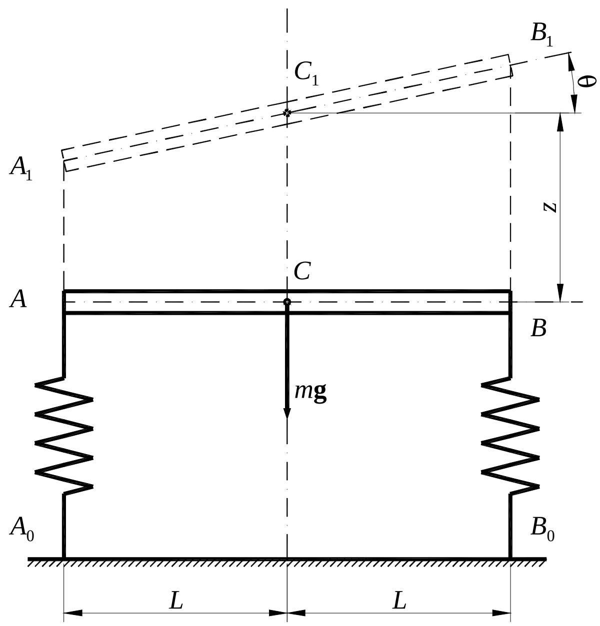

Figure 1. Mathematical model.

The model has two degrees of freedom, name them: the vertical displacement of the centre of weight, $C_1C = z$, and the rotation of angle θ about a horizontal axis that passes through the centre of weight, the rotation being small enough such that we can consider that the springs remain vertical.

We consider that the elastic force that appears in a spring derives from a potential in the form

$$W = \frac{k}{2} d^2 + \frac{\varepsilon}{p} d^p, \tag{1}$$

where $k > 0$, p is a natural number equal at least to 3, d is the elongation of the spring, and $\varepsilon \in \mathrm{R}$.

The equations of motion

The kinetic energy of the system is

$$T = \frac{1}{2} m\dot{z}^2 + \frac{1}{2} J\dot{\theta}^2, \quad (2)$$

and the potential energy has the expression

$$V = mgz + \frac{k}{2}(z - L\theta)^2 + \frac{\varepsilon}{p}(z - L\theta)^p + \frac{k}{2}(z + L\theta)^2 + \frac{\varepsilon}{p}(z + L\theta)^p . \quad (3)$$

One obtains the Lagrange equations [2]

$$m\ddot{z} + mg + 2kz + \varepsilon\left[(z + L\theta)^{p-1} + (z - L\theta)^{p-1}\right] = 0, \quad (4)$$

$$J\ddot{\theta} + 2kL^2\theta + \varepsilon L\left[(z + L\theta)^{p-1} - (z - L\theta)^{p-1}\right] = 0. \quad (5)$$

The equilibrium positions

These equilibrium positions are deduced from the equations [3], [4], [5]

$$2kz + \varepsilon\left[(z + L\theta)^{p-1} + (z - L\theta)^{p-1}\right] + mg = 0, \quad (6)$$

$$2kL\theta + \varepsilon\left[(z + L\theta)^{p-1} - (z - L\theta)^{p-1}\right] = 0. \quad (7)$$

Summing the equations (6) and (7), one obtains

$$2k(z + L\theta) + 2\varepsilon(z + L\theta)^{p-1} = -mg, \quad (8)$$

and by subtracting the same equations, we get

$$2k(z - L\theta) + 2\varepsilon(z - L\theta)^{p-1} = -mg. \quad (9)$$

Equating the relations (8) and (9), we deduce

$$2k(z + L\theta) + 2\varepsilon(z + L\theta)^{p-1} = 2k(z - L\theta) + 2\varepsilon(z - L\theta)^{p-1}, \quad (10)$$

wherefrom

$$2kL\theta = \varepsilon\left[(z - L\theta)^{p-1} - (z + L\theta)^{p-1}\right] \quad (11)$$

or, equivalently,

$$2kL\theta = -2\varepsilon L\theta\left[(z - L\theta)^{p-2} + (z - L\theta)^{p-3}(z + L\theta) + \ldots + (z + L\theta)^{p-2}\right]. \quad (12)$$

It is easy to prove, considering the cases $p-1$ odd and $p-1$ even, that the equation (11) leads to the unique solution $\theta = 0$.

With this value for θ for the equilibrium, the equation (7) becomes an identity, and the equation (6) leads us to

$$2\varepsilon z^{p-1} + 2kz + mg = 0. \quad (13)$$

Applying the Descartes theorem [7], the equation (13) has unique solution if and only if $\varepsilon > 0$ and $p-1$ is an odd number.

The stability of the equilibrium

Denoting $\xi_1 = z$, $\xi_2 = \theta$, $\xi_3 = \dot{z}$, $\xi_4 = \dot{\theta}$, the Lagrange equations (4) and (5) transform in a system of four first order non-linear differential equations

$$\dot{\xi}_1 = \xi_3 \text{g}, \; \dot{\xi}_2 = \xi_4, \; \dot{\xi}_3 = -\frac{2k\xi_1}{m} - \frac{\varepsilon}{m}\left[(\xi_1 + L\xi_2)^{p-1} + (\xi_1 - L\xi_2)^{p-1}\right] - g,$$
$$\dot{\xi}_4 = -\frac{2kL^2\xi_2}{J} - \frac{\varepsilon L}{J}\left[(\xi_1 + L\xi_2)^{p-1} - (\xi_1 - L\xi_2)^{p-1}\right]. \tag{14}$$

Denoting by f_i, $i = \overline{1,4}$, the functions defined by right-hand terms in the expressions (14) and by $j_{kl} = \dfrac{\partial f_k}{\partial \xi_l}$, $k, l = \overline{1,4}$, their partial derivatives, it results the characteristic equation [3], [4], [5], [6], [8]

$$\begin{vmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ j_{31} & j_{32} & -\lambda & 0 \\ j_{41} & j_{42} & 0 & -\lambda \end{vmatrix} = 0, \tag{15}$$

wherefrom we obtain

$$\lambda^4 - (j_{31} + j_{42})\lambda^2 + (j_{31}j_{42} - j_{41}j_{32}) = 0. \tag{16}$$

On the other hand, at the equilibrium $\xi_2 = 0$, $\xi_3 = 0$, $\xi_4 = 0$, such that it follows

$$j_{31} = -\frac{2k}{m} - \frac{2\varepsilon}{m}(p-1)\xi_1^{p-2}, \; j_{32} = 0, \; j_{41} = 0,$$
$$j_{42} = -\frac{2kL^2}{J} - \frac{2\varepsilon L^2}{J}(p-1)\xi_1^{p-2} \tag{17}$$

and the characteristic equation (16) becomes

$$\lambda^4 - (j_{31} + j_{42})\lambda^2 + j_{31}j_{42} = 0, \tag{18}$$

with the obvious solutions $\lambda_{1,2} = \pm\sqrt{j_{31}}$, $\lambda_{3,4} = \pm\sqrt{j_{42}}$.

The stability is assured if $j_{31} < 0$ and $j_{42} < 0$.

Let us observe that if p is an even number, $p \geq 4$, then it is aure that $j_{31} < 0$ and $j_{42} < 0$ and the equilibrium is stable, but not asymptotically stable.

The stability of the motion

Let be the deviations $\xi_i \mapsto \xi_i + \zeta_i$, $i = \overline{1,4}$, and the system in deviations

$$\dot{\zeta}_1 = \zeta_3, \tag{19a}$$

$$\dot{\zeta}_2 = \zeta_4, \tag{19b}$$

$$\dot{\zeta}_3 = -\frac{2k\zeta_1}{m} - \frac{\varepsilon}{m}\left[(\xi_1 + \zeta_1 + L\xi_2 + L\zeta_2)^{p-1} + (\xi_1 + \zeta_1 - L\xi_2 - L\zeta_2)^{p-1}\right] + $$
$$+ \frac{\varepsilon}{m}\left[(\xi_1 + L\xi_2)^{p-1} + (\xi_1 - L\xi_2)^{p-1}\right], \tag{19c}$$

$$\dot{\zeta}_4 = -\frac{2kL^2\zeta_2}{J} - \frac{\varepsilon L}{J}\left[(\xi_1 + \zeta_1 + L\xi_2 + L\zeta_2)^{p-1} - (\xi_1 + \zeta_1 - L\xi_2 - L\zeta_2)^{p-1}\right] + + \frac{\varepsilon L}{J}\left[(\xi_1 + L\xi_2)^{p-1} - (\xi_1 - L\xi_2)^{p-1}\right]. \quad (19d)$$

We shall limit ourselves to the case $p = 4$, when the system (19) becomes

$$\dot{\zeta}_1 = \zeta_3,\ \dot{\zeta}_2 = \zeta_4,$$
$$\dot{\zeta}_3 = -\frac{2k\zeta_1}{m} - \frac{\varepsilon}{m}\left[3(\xi_1 + L\xi_2)^2(\zeta_1 + L\zeta_2) + 3(\xi_1 - L\xi_2)^2(\zeta_1 - L\zeta_2)\right] + NLT_1, \quad (20)$$
$$\dot{\zeta}_4 = -\frac{2kL^2\zeta_2}{J} - \frac{\varepsilon L}{J}\left[3(\xi_1 + L\xi_2)^2(\zeta_1 + L\zeta_2) - 33(\xi_1 - L\xi_2)^2(\zeta_1 - L\zeta_2)\right] + NLT_2,$$

where NLT_1 and NLT_2 stand for the non-linear terms in ζ_1 and ζ_2.

The system (20) can be brought to the simpler form

$$\dot{\zeta}_1 = \zeta_3,\ \dot{\zeta}_2 = \zeta_4,\ \dot{\zeta}_3 = -\left[\frac{2k}{m} + \frac{6\varepsilon}{m}\left(\xi_1^2 + L^2\xi_2^2\right)\right]\zeta_1 - \frac{12\varepsilon}{m}\xi_1\xi_2 L\zeta_2 + NLT_1,$$
$$\dot{\zeta}_4 = -\frac{12\varepsilon L^2\xi_1\xi_2}{J}\zeta_1 - \left[\frac{2kL}{j} + \frac{6\varepsilon L}{J}\left(\xi_1^2 + L^2\xi_2^2\right)\right]L\zeta_2 + NLT_2. \quad (21)$$

One obtains again the characteristic equation (16) in which

$$j_{31} = -\left[\frac{2k}{m} + \frac{6\varepsilon}{m}\left(\xi_1^2 + L^2\xi_2^2\right)\right] < 0,\ j_{32} = -\frac{12\varepsilon}{m}\xi_1\xi_2 L,\ j_{41} = -\frac{12\varepsilon L^2\xi_1\xi_2}{J},$$
$$j_{42} = -\left[\frac{2kL}{j} + \frac{6\varepsilon L}{J}\left(\xi_1^2 + L^2\xi_2^2\right)\right]L < 0, \quad (22)$$

such that

$$j_{31} + j_{42} < 0,\ j_{31}j_{42} + j_{41}j_{32} > 0,\ \left(j_{31} - j_{42}\right) + 4j_{41}j_{32} > 0. \quad (23)$$

Denoting $\lambda^2 = u$ and writing the equation (16) in u, with the roots u_1 and u_2, it immediately results that $u_1 < 0$, $u_2 < 0$ and therefore the equation (16) has four pure imaginary roots.

In conclusion, the motion is stable, but not asymptotically stable.

Application

Let us consider the numerical case described by $m_1 = 1400\,\mathrm{kg}$, $L = 2\,\mathrm{m}$, $J = 900\,\mathrm{kgm}^2$, $k = 10^5\ \mathrm{N/m}$, $\varepsilon = 112000\ \mathrm{N/m}^3$, $g = 10\,\mathrm{m/s}^2$.

The equation (13) leads us to

$$224000z^3 + 200000z + 14000 = 0, \quad (24)$$

with the solution $z = -0.0694\,\mathrm{m}$.

In the linear case the solution is $z_l = -0.07\,\mathrm{m}$, very closed to the previous.

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