ANALELE UNIVERSITĂȚII
"EFTIMIE MURGU" REȘIȚA
ANUL XVII, NR. 1, 2010, ISSN 1453 - 7397

Syed Noh, Mostafa Abdalla, Waleed Fekry Faris

A Study of Annular Plate Buckling Problem with Tension Loaded at Inner Edge

The buckling problem of annular plate with only one edge (either inner or outer edge) constrained is formulated and presented in this paper. The boundary condition is simply supported inner edge - free outer edge. The results are compared between the FEM results obtained using ABAQUS and results reported in literature. The results shows good agreement for critical buckling load, mode shapes and stress distributions.

Keywords: *piezoelectric, annular, buckling, methods of multiple scales*

1. Introduction

Plate buckling problem has been a classical problem in solid mechanics. There are great many literature and books that deals with plate buckling problem. Many of the annular plate buckling are formulated with the in-plane condition of both inner edge and outer edge are the similar, i.e. at both edge the plate is allowed to move in-plane and loaded with the same magnitude of load. This case will lead to a uniform stress distribution throughout the plate. This will lead to governing equation which the differential equation that could be solved analytically. However, for an annular plate with different in-plane condition, the stresses are varied with the radius which complicate the governing equation and the solution is not analytically available. Mansfield [1] reported on the plate buckling analysis with considering such stresses distribution throughout the plate. However, he did the buckling analysis for infinite annular plate which simplified the governing buckling equation and the analytical solution for such problem is available. Recently, Coman and Haughton [2-3] have presented a reports on annular plate with simply supported inner edge and free outer edge and the tensile load is applied at the inner edge. They used compound matrix methods to solve the buckling governing equation. The intention of this paper is to compare a finite element analysis results with the results obtain by Coman and Haughton [2]. This will provide the basis for comparison for asymptotic solution to be obtained later.

2. Problem Statement

Figure 1 shows the annular plate with tensile load applied at the inner edge. In the present analysis, the cylindrical polar coordinate is adopted. The origin of the coordinate is located at the centre of the annular plate. The inner and outer radius, and thickness of the annular plate are denoted by r_o, r_i and h, respectively.

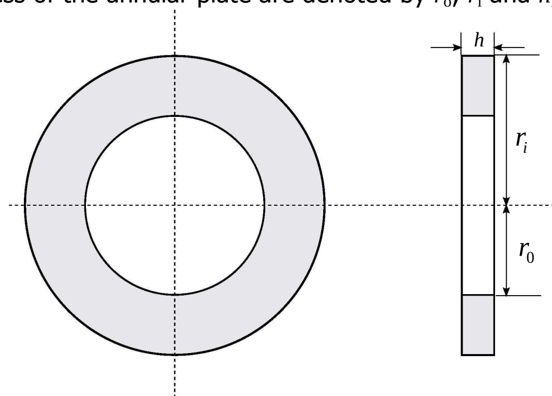


Figure 1. The geometry of an annular plate.

The plate formulation is formulated using the Classical Plate Theory (CPT). For such a plate, the strains are defined as

$$e_{11} = \frac{\partial u}{\partial r} \text{ , } e_{22} = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } e_{12} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \tag{1}$$

where u, v, and w, are the displacement in r, θ and z direction respectively. The corresponding stresses obeys the classical Hooke's Law which can be written as

$$\sigma_{11} = \frac{E\left(e_{11} + \mu e_{22}\right)}{1-\mu^2} \text{ , } \sigma_{22} = \frac{E\left(e_{22} + \mu e_{11}\right)}{1-\mu^2} \text{ and } \sigma_{12} = \frac{E e_{12}}{2\left(1+\mu\right)} \tag{2}$$

where E and μ are the Young Modulus and Poisson's ratio respectively.

The equilibrium equation for annular or circular plate can be written as

$$-\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rN_{11}\right) + \frac{\partial N_{12}}{\partial \theta} - N_{22}\right] = 0\text{ ,} \tag{3}$$

$$-\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rN_{12}\right) + \frac{\partial N_{22}}{\partial \theta} + N_{22}\right] = 0\text{, and} \tag{4}$$

$$
-\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rM_{11}\right)-\frac{\partial M_{22}}{\partial r}+\frac{1}{r}\frac{\partial^2 M_{22}}{\partial\theta^2}+\frac{2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial M_{12}}{\partial\theta}\right)\right.
$$

$$
\left.+\frac{\partial}{\partial r}\left(rN_{11}\frac{\partial w_0}{\partial r}+N_{12}\frac{\partial w_0}{\partial\theta}\right)+\frac{1}{r}\frac{\partial}{\partial\theta}\left(N_{22}\frac{\partial w_0}{\partial\theta}+rN_{12}\frac{\partial w_0}{\partial r}\right)\right]=0 \qquad (5)
$$

where the forces, N_{ij} and moments, M_{ij} are defined as the integration of stresses with respect to the thickness and the integration of the product of the stresses and thickness variable with respect of the thickness. For a linear problem, these equations are uncoupled and can be solved independently. However, for such a problem that being consider in this study, the equations are coupled but may be linearized.

In this study the pre-buckling state is assumed to be axisymmetric such that $u_0 = u_0(r)$ and $v = 0$. By noting this assumption, from Equation (1), the strains becomes

$$
e_{11}=\frac{du}{dr},\ e_{22}=\frac{u}{r}\ \text{and}\ e_{12}=0\,. \qquad (6)
$$

Applying Equation (6) into the constitutive equation lead to

$$
\sigma_{11}=\frac{E}{1-\mu^2}\left(\frac{du}{dr}+\mu\frac{u}{r}\right)\ ,\ \sigma_{22}=\frac{E}{1-\mu^2}\left(\frac{u}{r}+\mu\frac{du}{dr}\right)\ \text{and}\ \sigma_{12}=0 \qquad (7)
$$

where the forces may be determined as

$$
N_{11}=\frac{2Eh}{1-\mu^2}\left(\frac{du}{dr}+\mu\frac{u}{r}\right)\ \text{and}\ N_{22}=\frac{2Eh}{1-\mu^2}\left(\frac{u}{r}+\mu\frac{du}{dr}\right). \qquad (8)
$$

Subtituting this expressions into Equation (3) lead to the Eulerian differential equation;

$$
r^2\frac{d^2u}{dr^2}+r\frac{du}{dr}-u=0 \qquad (9)
$$

which the general solution can be expressed as

$$
u=\frac{C_1}{r}+C_2 r \qquad (10)
$$

where C_1 and C_2 are determined by the boundary conditions. In the present study the boundary condition is set as simply supported at inner edge and free at outer edge while a uniform radial load, N_0 is applied and traction free at outer edge. After determination of the constants C_1 and C_2 under this conditions, the loads can now be expressed as

$$
N_{11}=\frac{N_0 r_i^2 r_o^2}{r_o^2-r_i^2}\left(\frac{1}{r^2}-\frac{1}{r_o^2}\right)\ \text{and}\ N_{22}=\frac{N_0 r_i^2 r_o^2}{r_o^2-r_i^2}\left(\frac{1}{r^2}+\frac{1}{r_o^2}\right). \qquad (11)
$$

The buckling equation may be obtained from Equation (5) in terms of deflection, w and the loads

$$D\nabla^4 w - N_{11}\frac{\partial^2 w}{\partial r^2} - N_{22}\frac{1}{r}\left(\frac{\partial w}{\partial r} + \frac{1}{r}\frac{\partial^2 w}{\partial \theta^2}\right) = 0 \tag{12}$$

where $D=Eh^3/12(1-\mu^2)$ is the bending stiffness and N_{11} and N_{22} is defined as in Equation (11). Solving Equation (12) will lead to the desired buckling load. The analytical solution for Equation (12) is not available and it is customary to solve it through perturbation method. In this particular paper, the results for similar condition reported in [2] is compared with the finite element analysis. This comparison will provide the basis for validation of asymptotic solution obtained through the perturbation methods.

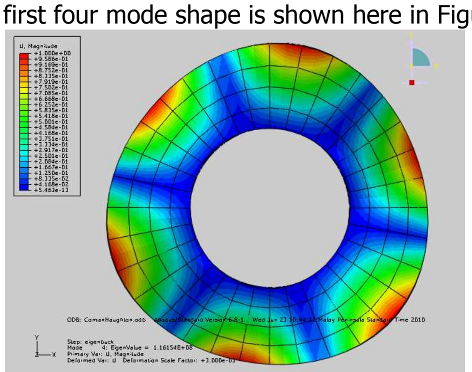
3. Results and Discussions

An annular plate with inner radius r_i = 7.5 mm, outer radius r_o = 15 mm and thickness, and h = 0.3 mm is selected as example for the analysis. The plate is an alloy plate with Young modulus, E = 110 GPa and Poisson ratio μ = 0.34. The FEM analysis is done by using ABAQUS 6.8. The plate is model using a 20-node, three-dimensional, solid elements (C3D20R). The results comparison between results reported from [2] and the FEM result obtained via ABAQUS 6.8 are shown in Table 1.

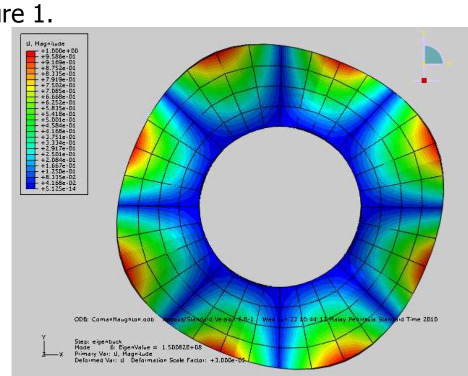
Table 1.

Mode	Critical Buckling Load, N_0 [N/m]		Difference [%]
	(a) Literature [2]	(b) FEM	
1	3.47E+04	3.49E+04	0.401722
2	4.57E+04	4.50E+04	-1.532652
3	5.68E+04	5.39E+04	-5.229001
4	6.58E+04	6.18E+04	-6.435964
5	8.96E+04	8.36E+04	-7.203542

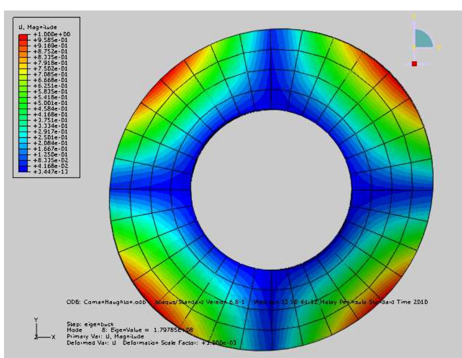
The results shows a good agreement between the one that calculated by ABAQUS and the one reported by Coman and Haughton [2]. As expected, the error grew as the mode increase. This is due to the assumed mode shape used in the formulation of the finite element analysis. The results obtained by finite element will be close to analytical solution if the assumed mode shapes close to the true mode shapes. Furthermore, the mode shapes predicted by the ABAQUS also confirmed the mode shape reported by Coman and Haughton [2]. As example, the first four mode shape is shown here in Figure 1.



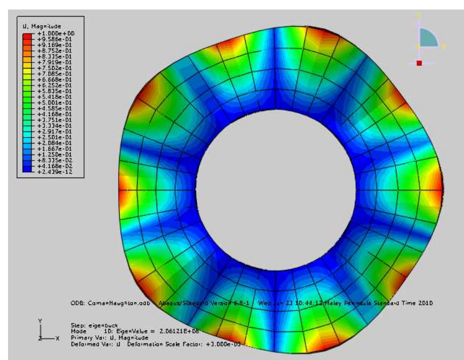
a. Mode 1



b. Mode 2



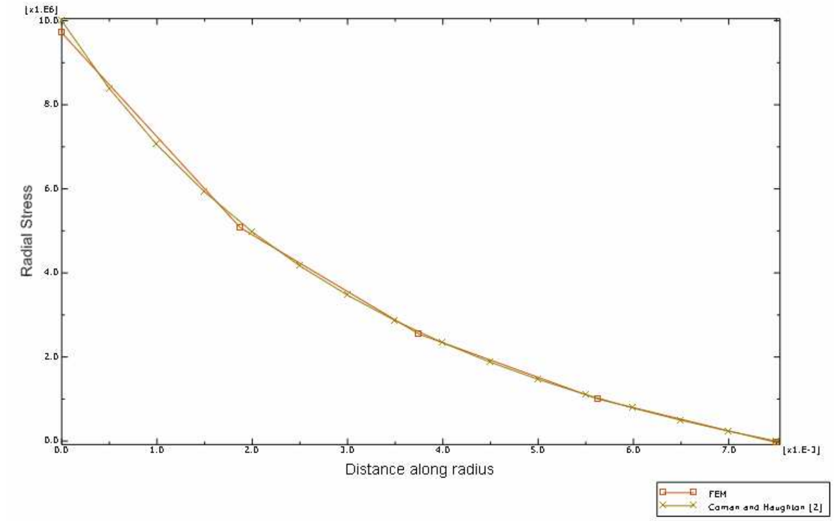
c. Mode 3



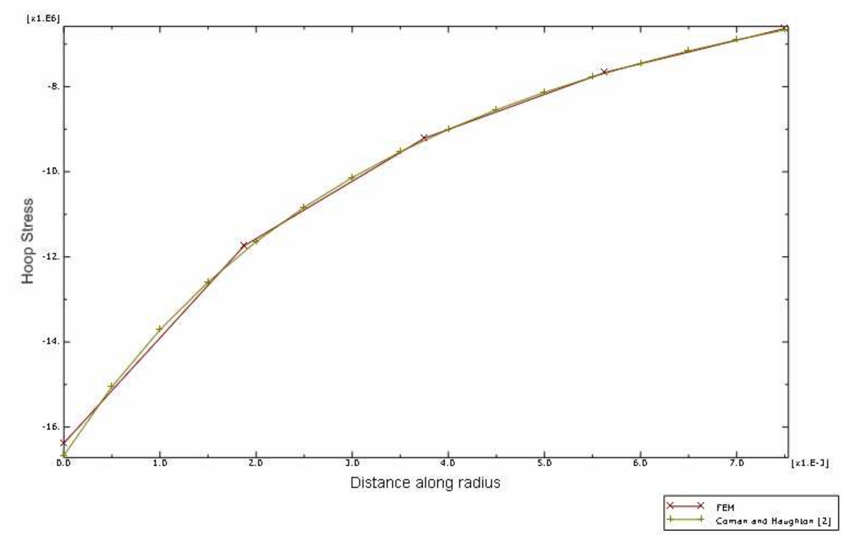
d. Mode 4

Figure 1. The mode shape for mode 1 to 4 obtained from ABAQUS

As mentioned earlier in the introductions, the stress distribution for an annular plate where only one of its edges (in the present case is the inner edge) is loaded is not uniform and it is a function of its radius. The stress distribution throughout the annular plate is given by the equation (11) divided by its thickness. The stress distribution obtained by ABAQUS give an excellent agreement with the analytical stress distribution. The comparison between these two distributions is shown in Figure 2. The magnitude of the tension loaded at the inner edge is 3 kN/m.



(a.)



(b.)

Figure 2. (a.) The radial stress distribution as a function of radius (b.) The hoop stress distribution as function of radius

4. Conclusion

The results of finite element analysis of annular buckling problem, which only the inner edge is loaded with tension load, has been compared to the results reported by Coman and Haughton [2]. The critical buckling load, mode shapes and the stress distribution obtained by ABAQUS have shown good agreement with the one that reported by Coman and Haughton [2]. In addition, it is noticed that the error calculated for critical buckling load increased as the mode increase. It is believed that this result will be the sufficient basis for validation of the asymptotic result that will be determined in the future.

References

[1] Mansfield E. H., *On the buckling of an annular plate,* Quarterly Journal of Mechanical Applied Mathematics, vol. 13., no. 1. 16-23.
[2] Coman C. D. and Haughton D. M., *On some approximation methods for tensile instabilities of thin annular plates*, Journal of Engineering Mathematics, vol. 56, 79-99.
[3] Coman C. D. and Haughton D. M., *Localized wrinkling instabilities in radially stretched annular thin films*, Acta Mechanica, vol. 185, 179-200.

Addresses:-

- Syed Noh, International Islamic University Malaysia, P.O. Box 10 Kuala Lumpur, syednoh@iium.edu.my
- Assoc. Prof. Dr. Waleed Fekry Faris, International Islamic University Malaysia, P.O. Box 10 Kuala Lumpur, waleed@iium.edu.my
- Asst. Prof. Dr. Mostafa Abdalla, Delft University of Technical, Kluyverweg, Delft, Netherlands, m.m.abdalla@tudelft.nl