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Economic Models with Discrete Time. State of Equilibrium

In this paper is present the economic models with discrete time. An economic model is followed of numerical program in Matlab, which exemplify the state of equilibrium.

***Keywords**: economic model, stability, equilibrium state*

1. Preliminaries and definitions
Linear equations with discrete time

This paper present the solutions of linear equations, which depends at form of free term.

Thus considers the equation with discrete time of form

$$ax_{t+2} + bx_{t+1} + cx_t = f_t, \qquad (1)$$

where a,b,c is real values. For this equation we search the particular solutions which depend at form of sequence f_t. Thus we have more cases.

I. In this case considers the constant sequence f_t, thus we have:

a. if $a + b + c \neq 0$ then exist a constant solution of form

$$\hat{x} = \frac{f}{a + b + c} .$$

b. we have $a + b + c = 0$ only if the characteristic equation of equation (1)

$$aq^2 + bq + c + 0$$

have the root q=1.

We suppose that characteristic equation have root q=1. Then result c= -(a+b) and recurrent equation is the form

$$ax_{t+2} + bx_{t+1} - ax_t - bx_t = f . \qquad (2)$$

We denote $x_{t+1} - x_t = y_t$ and we have the relation

$$x_{t+2} - x_t = y_{t+1} + y_t .$$

Substituted this equation in equation (2) and result the relation

$$ay_{t+1} + (a+b)y_t = f\ . \qquad (3)$$

For this equation we have the constant solution of form

$$\hat{y} = \frac{f}{2a+b},$$

with condition $2a + b \neq 0$, i.e. the characteristic equation of equation (3) have the multiple root 1. In this case result

$$x_{t+1} - x_t = \frac{f}{2a+b},$$

i.e. we have the solution of form

$$x_t = t\frac{f}{2a+b} + x_0\ .$$

c. if the root q=1 is multiple root of characteristic equation then

$$a+b+c=0 \quad \text{and} \quad 2a+b=0$$

from we have c=a and b= -2a. Thus the equation (1) will be of form

$$a(x_{t+2} - 2x_{t+1} + x_t) = f\ .$$

If denote $x_{t+1} - x_t = y_t$ result the equation

$$y_{t+1} - y_t = \frac{f}{a}.$$

From this relation we have

$$y_t = t\frac{f}{a} + y_0,$$

and come to the point of above notation, result the relation

$$x_{t+1} - x_t = t\frac{f}{a} + y_0\ .$$

From this relation result the general solution of equation (1)

$$x_t = x_0 + t(x_1 - x_0) + \frac{t(t-1)}{2}\frac{f}{a}\ .$$

II. In this case f_t is a polynomial function of t. Suppose that the polynomial function is the form $f_t = At + B$. This we have three cases

a. if $a+b+c \neq 0$ then for equation (1) we search a solution of form

$$x_t = \alpha t + \beta\ . \qquad (4)$$

Substituted this relation in equation (1) and result

$$a(\alpha t + 2\alpha + \beta) + b(\alpha t + \alpha + \beta) + c(\alpha t + \beta) = At + B\ .$$

Thus we obtain the coefficients of equation (4)

$$\alpha = \frac{A}{a+b+c}, \quad \beta = \frac{B}{a+b+c} - \frac{(2a+b)A}{(a+b+c)^2}\ .$$

b. if $a+b+c=0$ and $2a+b \neq 0$, then we search a solution of form

$$x_t = \alpha t^2 + \beta t\ .$$

Substituted this solution, result the relation

$$a\left[\alpha(t+2)^2+\beta(t+2)\right]+b\left[\alpha(t+1)^2+\beta(t+1)\right]+c\left[\alpha t^2+\beta t\right]=At+B\,.$$

Thus we obtain the system of equations

$$\begin{cases}\alpha(a+b+c)=0\\ 2(2a+b)\alpha+(a+b+c)\beta=A\\ 4a\alpha+2a\beta+b\alpha+b\beta=B\end{cases}$$

from where result that the coefficients of general solution is at form

$$\alpha=\frac{A}{2(2a+b)} \quad \text{and} \quad \beta=\frac{B-(4a+b)\alpha}{2a+b}$$

c. if the coefficients a,b şi c respectively, prove the relations

$$a+b+c=0 \quad \text{and} \quad 2a+b=0$$

then we search a solution of form

$$x_t=\alpha t^3+\beta t^2\,.$$

Again, from substitution result

$$a\left[\alpha(t+2)^3+\beta(t+2)^2\right]+b\left[\alpha(t+1)^3+\beta(t+1)^2\right]+c\left[\alpha t^3+\beta t^2\right]=At+B$$

and obtain the system of equations

$$\begin{cases}\alpha(a+b+c)=0\\ 3\alpha(2a+b)+(a+b+c)\beta=0\\ 12a\alpha+4a\beta+3b\alpha+2b\beta=A\\ 8a\alpha+4a\beta+b\alpha+b\beta=B\end{cases}$$

from where obtain the coefficients of solution

$$\alpha=\frac{A}{3(4a+b)} \quad \text{and} \quad \beta=\frac{B-(8a+b)\alpha}{4a+b}$$

for $4a+b\neq 0$.

III. Suppose that $f_t=\beta\alpha^t$, where β is constant. We search a solution of same form with free term, i.e.

$$x_t=\gamma\alpha^t\,.$$

Substituted this solution in equation (1) result the relation

$$a\gamma\alpha^{t+2}+b\gamma\alpha^{t+1}=\beta\alpha^t$$

and obtain

$$\left(a\alpha^2+b\alpha+c\right)\gamma=\beta\,.$$

Thus result

$$\gamma=\frac{\beta}{a\alpha^2+b\alpha+c}\,.$$

a. If α don't be the root of characteristic equation, i.e. $a\alpha^2+b\alpha+c\neq 0$, then the coefficient of solution is de above form, i.e. we have the solution of form

$$x_t = \frac{\beta}{a\alpha^2 + b\alpha + c}\alpha^t.$$

b. If α is the root of characteristic equation then we search a solution of form

$$x_t = \gamma t \alpha^t.$$

Again from substitution result the equation

$$a\gamma(t+2)\alpha^{t+2} + b\gamma(t+1)\alpha^{t+1} + c\gamma t\alpha^t = \beta\alpha^t.$$

Thus we obtain the system

$$\begin{cases} a\alpha^2 + b\alpha + c = 0 \\ \gamma\alpha(2a\alpha + b) = \beta \end{cases}$$

from where result

$$\gamma = \frac{\beta}{\alpha(2a\alpha + b)}, \text{ for } 2a\alpha + b \neq 0.$$

c. If α is multiple root of characteristic equation. i.e. if

$$a\alpha^2 + b\alpha + c = 0 \text{ and } 2a\alpha + b = 0.$$

Then we search a solution of form

$$x_t = \gamma t 2\alpha^t.$$

And in the same way obtain

$$\gamma = \frac{\beta}{\alpha(4a\alpha + b)}, \quad \forall\alpha \in R.$$

2. Samuelson-Hicks mathematic model with discrete time

In this paragraph we present the Samuelson-Hicks mathematic model with discrete time. In this model we have the notation:

- Y_t - is national product in moment t, allocate for consumption and investment;
- I_t - is investment of moment t;
- A_t - is autonomous component, independent of income, and of national product respectively;
- C_t - is consumption in moment t.

We suppose that investment in moment t depends of national product variations in previous moments, i.e.

$$Y_{t-1} - Y_{t-2} = \frac{1}{\nu} I_t \quad . \qquad (5)$$

And the national consumption in period t depend linear at national income in moment t-1 and national income in moment t-2, respectively. Thus we have the relation

$$C_t = \gamma + c_1 Y_{t-1} + c_2 Y_{t-2} \qquad (6)$$

where $0<c_1<1$, $0<c_2<1$ and $c_1+c_2=c<1$.

Substituted the relations (5) and (6) in relation (2) we obtain

$$Y_t - (v + c_1)Y_{t-1} + (v - c_2)Y_{t-2} = A_t + \gamma . \qquad (7)$$

Thus we have two cases, which depend on the form of autonomous investment A_t.

I. If the autonomous investment A_t is constant, then the value of equilibrium of national consumption

$$\hat{Y} = \frac{A+\gamma}{1-c} = \frac{A+\gamma}{s}$$

Denote $v - c_2 = w$, $y_t = Y_t - \hat{Y}$ and result the relation

$$y_{t+1} - (1 + w - s)\, y_t + w\, y_{t-2} = 0 .$$

For characteristic equation we have

$$\Delta = (1 + w - s)^2 - 4w = w^2 + [2(1-s) - 4]w + (1-s)^2 =$$
$$= w^2 - 2(s+1)w + (1-s)^2.$$

and the roots is

$$w_{1,2} = 1 + s \pm \sqrt{(1+s)^2 - (1-s)^2} = 1 + s \pm 2\sqrt{s} = (1 \pm \sqrt{s})^2$$

Thus we have more cases, which depend of value w.

a. If the parameter w satisfy

$$(1-\sqrt{s})^2 < w < (1+\sqrt{s})^2$$

then $\Delta < 0$. Thus the evolution of this model will be oscillating. This oscillate is brake for w<1, and explosive for w>1.

b. If we have

$$w < (1-\sqrt{s})^2$$

then $\Delta > 0$, and the roots is real value. Result that

$$-1 < w < 1 .$$

This condition ensure that the roots of characteristic equation is in (-1,1), and the solution of equation (7) will be converge on equilibrium state.

c. If $w > (1+\sqrt{s})^2$ then the roots of characteristic equation is real end positives. Because the product of the roots is major that 1, result the solutions of equation (7) converge on infinite.

II. Now we have that the autonomous investments A_t is given by

$A_t = I^a r^t$, where I^a is a constant.

Consider that the investments on moment t is given by

$$I_t = I_t^{ind} + A_t \qquad (8)$$

where I_t^{ind} is investments induced in the period t.

The relations which determine this model is

$$C_t = c_0 + c_1 Y_{t-1} + c_2 Y_{t-2} \qquad (9)$$

$$I_t^{ind} = \beta\,(Y_{t-1} - Y_{t-2}) \quad \text{where } \beta \geq 0,\ c_0 \geq 0 \text{ and } 0 < c_1 + c_2 < 1. \tag{10}$$

We start on the equilibrium equation that is the form

$$Y_t = C_t + I_t \tag{11}$$

Substituted the above relations in the equilibrium equation we have

$$Y_t - (c_1 + \beta)Y_{t-1} - (c_2 - \beta)Y_{t-2} = c_0 + I^a r^t \qquad 12)$$

The solution of this equation is

$$Y_t = Y_t^0 + Y^{c_0} + Y_t^r \tag{13}$$

where

$$Y^{c_0} = \frac{c_0}{1 - c_1 - c_2}, \qquad Y_t^r = \frac{r^2 I^a}{r^2 - (c_1 + \beta)r - (c_2 - \beta)} r^t$$

and Y_t^0 is solution of homogeneous equation

$$Y_t - (c_1 + \beta)Y_{t-1} - (c_2 - \beta)Y_{t-2} = 0\,. \tag{14}$$

If we have the condition $r \in (0,1]$ then the solution (13) will be convergent. Observe that Y_t is convergent on equilibrium solution Y^{c_0} ($Y^{c_0} + I^a$ respectively, if r=1) only if Y_t^0 is convergent on 0.

The characteristic equation of relation (14) is given by

$$p(\lambda) = \lambda^2 - (c_1 + \beta)\lambda + \beta - c_2\,.$$

I. If $\beta > c_2$, we have tree cases which depend of Δ.

a. If $\Delta > 0$, i.e.

$$c_2 > \beta - \frac{(c_1 + \beta)^2}{4},$$

Result that characteristic equation has two roots, λ_1 and λ_2. Because

$$\lambda_{min} = \frac{c_1 + \beta}{2} > 0$$

and the minim value is $f(\lambda_{min}) = -\frac{\Delta}{4}$ <0, result that the roots is less than 1 (major than 1, respectively) only if λ_{min} is less than 1 (major than 1, respectively).
The solution of equation (14) is given by

$$Y_t^0 = m_1\lambda_1^t + m_2\lambda_2^t\ . \tag{15}$$

Thus the solution of equation (12) is

$$Y_t = m_1\lambda_1^t + m_2\lambda_2^t + \frac{c_0}{1 - c_1 - c_2} + \frac{r^2 I^a}{r^2 - (c_1 + \beta)r - (c_2 - \beta)} r^t \tag{16}$$

Accordingly the solution

$$Y_t^0 \xrightarrow{t \to \infty} 0,$$

if and only if the roots is less than 1, i.e. $c_1 + \beta < 2$.

Thus the solution (16) will be convergent on equilibrium solution Y^{c_0}, and $Y^{c_0} + I^a$ respectively, if r=1. If don't have this, the solution will be convergent on infinite, for $t \to \infty$.

Now we present a Matlab program, which show how the solution converges on equilibrium solution Y^{c_0}. Consider the initial values

```
Y0=100000;
Y1=200000;
Ia=20000;
c0=50000;
c1=0.2;
c2=0.75;
beta=1.3;
r=[0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1]
t=[0 100 200 300 400 500 600 700 800 900 1000]
c=[1 -(c1+beta) beta-c2]
l=roots(c)
Yc0=c0/(1-c1-c2)
a=[1 1;log(l(1)) log(l(2))]
b=[100000;200000]
m=a\b
```

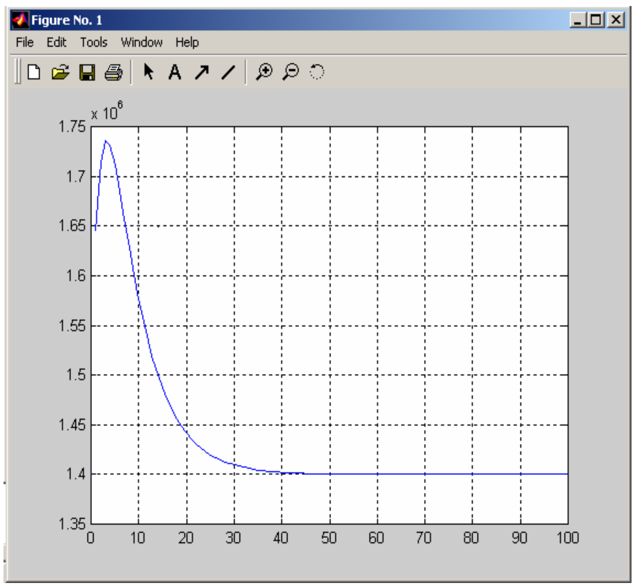

Figure 1. The graphic of solution Y_t which converges on equilibrium solution Y^{c_0}

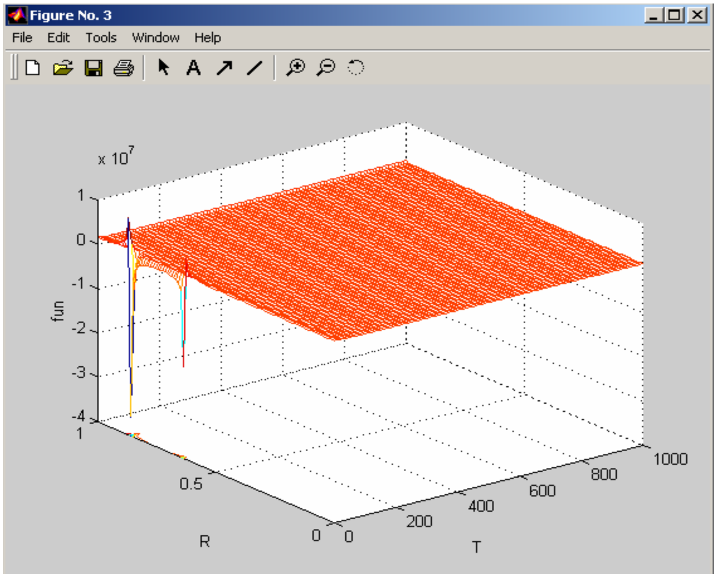

Figure 2. Evolution of solution Y_t

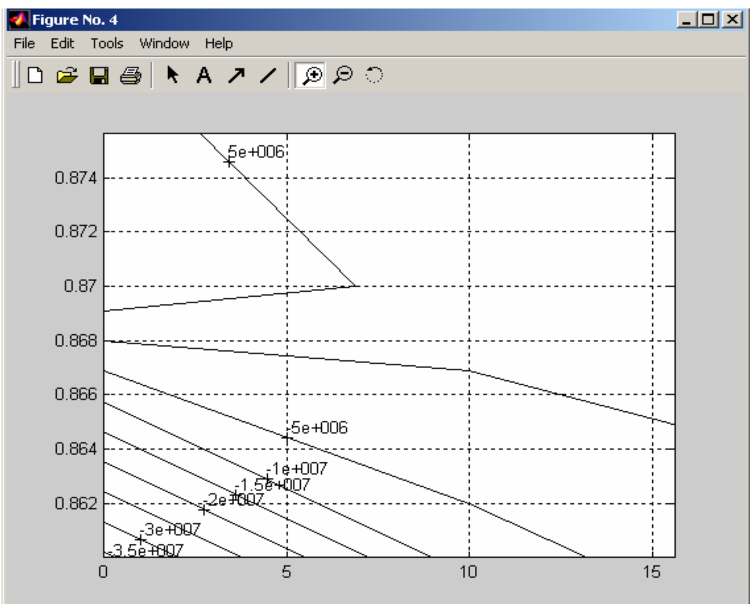

Figure 3. The values of solution Y_t, for the initial values considerate

b. If $\Delta = 0$, then $c_2 = \beta - \frac{(c_1+\beta)^2}{4}$,

and we have

$$\lambda = \lambda_1 = \lambda_2 = \frac{\beta + c_1}{2}.$$

In this case the solution of equation (14) will be

$$Y_t^0 = (m_1 + m_2 t)\lambda^t \quad , \qquad (17)$$

and in consequence $Y_t^0 \xrightarrow{t\to\infty} 0$, only if the roots is less than 1, i.e. $c_1 + \beta < 2$. In the other way, the solution will be convergent on infinite, for t→∞.

c. if $\Delta < 0$, and $c_2 < \beta - \frac{(c_1+\beta)^2}{4}$,

then the roots don't is real numbers, and will be given of

$$\lambda_{1,2} = r(\cos\theta \pm i\sin\theta).$$

Then the solution of (14), Y_t^0 will be

$$Y_t^0 = (\beta - c_2)^t (m_1 \cos t\theta + m_2 \sin t\theta). \qquad (18)$$

Denote $R = \beta - c_2$.

a) if R <1 then the solution $Y_t^0 \to 0$ for $t \to \infty$.

Thus the solution Y_t of equation (12) will be oscillated on equilibrium value.

b) if R=1 then

$$Y_t^0 = m_1 \cos t\theta + m_2 \sin t\theta$$

who oscillate around the origin. Thus the solution Y_t oscillated around origin.

c) if R >1 then $Y_t^0 \to \infty$ for $t \to \infty$.

Thus the solution Y_t present a increase oscillation on infinite.

II. In this case β=c_2, and the characteristic equation is

$\lambda^2 - (c_1+\beta)\lambda = 0$, and the roots is $\lambda_1 = 0$, $\lambda_2 < 1$.

Thus the solution of homogenous equation is

$$Y_t^0 = m_1\lambda_1^t + m_2\lambda_2^t = m_2(c_1+c_2)^t.$$

and $Y_t^0 \xrightarrow{t\to\infty} 0$.

In accordingly, the solution Y_t of equation (12) will be convergent on equilibrium state.

III. Now we have β<c_2, and the characteristic equation have a root λ_1 in (0,1), and a root is negative λ_2. Thus result that $|\lambda_2| < 1$.

From this, the solution of homogenous equation is

$$Y_t^0 = m_1\lambda_1^t + m_2\lambda_2^t.$$

Thus we have

$$Y_t^0 \xrightarrow{t\to\infty} 0,$$

and the solution Y_t of equation (12) converge on equilibrium solution.

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