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Stress and Strain of Solid Shafts in Interference Fit Couplings

Solid shaft is a common case in technical practice including interference fit couplings. Thick walled cylinders theory, which is used in interference fits computation, leads to wrong values of stress in case of solid shafts at the inner side. A numerical simulation using smaller and smaller ratios of the radii demonstrate a tendency towards stress homogeneity. Particularly along the axis of the shaft, the theory of stress under axial – symmetrical loading, analytically validates the numerical data. Practical consequences are very important as maximum admitted pressure is computed on the basis of equivalent stress. New checking formulae are proposed for interference fit couplings, containing the specific case of the solid shaft.

Keywords: *stress, strain, solid shafts, interference, fit couplings*

1. Introduction

In most technical applications, shafts are solid. In interference fit couplings, the solid shaft is approached as a particular case of thick walled cylinders (the specific feature is $r_1 = 0$). The issue is how valid the thick walled cylinders theory proves for this limit case.

Using the citied theory, for the shaft, which is loaded only with exterior pressure, p, and using the notation k for the ratio of interior/exterior radius ($k=r_1/r_2$), the stresses and radial displacement are [1]:

$$\sigma_r = \frac{-pr_2^2}{r_2^2 - r_1^2} + \frac{pr_1^2 r_2^2}{r_2^2 - r_1^2} \cdot \frac{1}{r^2}, \tag{1}$$

$$\sigma_t = \frac{-pr_2^2}{r_2^2 - r_1^2} - \frac{pr_1^2 r_2^2}{r_2^2 - r_1^2} \cdot \frac{1}{r^2}, \tag{2}$$

$$u = \frac{1-\nu}{E} \cdot \frac{-pr_2^2}{r_2^2 - r_1^2} \cdot r + \frac{1+\nu}{E} \frac{pr_1^2 r_2^2}{r_2^2 - r_1^2} \cdot \frac{1}{r}. \tag{3}$$

At the outer surface of the shaft, where $r = r_2$, results:

$$\sigma_r(r_2) = -p, \tag{4}$$

$$\sigma_t(r_2) = -p \cdot \frac{1+k^2}{1-k^2}, \tag{5}$$

$$u(r_2) = -\frac{pr_2}{E}\left(\frac{1+k^2}{1-k^2} - \nu\right). \tag{6}$$

At the inner surface of the shaft, where $r = r_1$, one gets:

$$\sigma_r(r_1) = 0, \tag{7}$$

$$\sigma_t(r_1) = -\frac{2p}{1-k^2}, \tag{8}$$

$$u(r_1) = -\frac{2pr_1}{E(1-k^2)}. \tag{9}$$

For the solid shaft, where $r_1 = 0$ (along the axis of the component), the above equations become:

- outer surface of the solid shaft ($r = r_2$):

$$\sigma_r = -p, \tag{10}$$

$$\sigma_t = -p, \tag{11}$$

$$u = -\frac{pr_2}{E}(1-\nu), \tag{12}$$

$$\sigma_T = p, \tag{13}$$

$$\sigma_M = p, \tag{14}$$

where σ_T is equivalent Tresca stress:

$$\sigma_T = max(|\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_3 - \sigma_1|) \tag{15}$$

and σ_M symbolizes equivalent von Mises stress:

$$\sigma_M = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}, \tag{16}$$

admitting the convention:

$$\sigma_1 < \sigma_2 < \sigma_3. \tag{17}$$

For the case under study: $\sigma_1 = \sigma_2 = -p$, $\sigma_3 = 0$.

- inner surface of the solid shaft ($r = r_1 = 0$):

$$\sigma_r = 0, \tag{18}$$

$$\sigma_t = -2p, \tag{19}$$

$$u = 0. \tag{20}$$

Normal principal stresses are:

$$\sigma_1 = -2p,\ \sigma_2 = \sigma_3 = 0, \tag{21}$$

and equivalent stress results:

$$\sigma_T = 2p, \tag{22}$$

$$\sigma_M = 2p. \tag{23}$$

2. Numerical analysis

An analysis of radial and tangential stress course, in relationship with the radius, was accomplished. Decreasing values of the ratio k were taken into account, downwards to a lower boundary of $k = 0$.

To make the demonstration clear and to assure general character of results, normalized values of pressure and outer radius were assumed. Hence, in mathematical modeling of stresses, in table 1, $p = 1$ and $r_2 = 1$ are emphasized on the first line. The shape of curves is alike no matter what numbers are taken for p and r_2. Five values of ratio k were used in computation (0.5, 0.3, 0.1, 0.01 and 0.001). They correspond to ratios of outer radius/inner radius equal to 2, 3.33, 10, 100 and 1000. The latter two values are big enough to go beyond the thick walled cylinders concept and to model a body geometrically closer to solid shafts.

Table 1.

p = 1, r₂ = 1							
k	r_2/r_1	r_1	r	σ_r [-]	σ_t [-]	σ_T [-]	σ_M [-]
0.5	2	0.5	0.6	-0.4074	-2.2593	2.2593	2.0856
			0.7	-0.6530	-2.0136	2.0136	1.7794
			0.8	-0.8125	-1.8542	1.8542	1.6098
			0.9	-0.9218	-1.7449	1.7449	1.5119
			1	-1	-1.6667	1.6667	1.4530
0.3	3.33	0.33	0.44	-0.5880	-1.6098	1.6098	1.4108
			0.58	-0.8049	-1.3929	1.3929	1.2112
			0.72	-0.9081	-1.2897	1.2897	1.1475
			0.86	-0.9652	-1.2326	1.2326	1.1230
			1	-1	-1.1978	1.1978	1.1122
0.1	10	0.1	0.28	-0.8813	-1.1389	1.1389	1.0345
			0.46	-0.9624	-1.0578	1.0578	1.0135
			0.64	-0.9854	-1.0348	1.0348	1.0110
			0.82	-0.9951	-1.0251	1.0251	1.0104
			1	-1	-1.0202	1.0202	1.0103
0.01	100	0.01	0.208	-0.9978	-1.0024	1.0024	1.0001
			0.406	-0.9995	-1.0007	1.0007	1.0001
			0.604	-0.9998	-1.0004	1.0004	1.0001
			0.802	-0.9999	-1.0003	1.0003	1.0001
			1	-1	-1.0002	1.0002	1.0001
0.001	1000	0.001	0.2008	-1.0000	-1.0000	1.0000	1.0000

			0.4006	-1.0000	-1.0000	1.0000	1.0000
			0.6004	-1.0000	-1.0000	1.0000	1.0000
			0.8002	-1.0000	-1.0000	1.0000	1.0000
			1	-1	-1	1.0000	1.0000

For each k ratio, the radial, tangential and equivalent Tresca and von Mises stresses were computed in five points of the current radius. The calculus is based on thick walled cylinders theory (rel. 1 and 2).

In figures 1…4 are drawn the families of curves $\sigma_r(r)$, $\sigma_t(r)$, $\sigma_T(r)$ and $\sigma_M(r)$ in relationship with the current radius *r* and *k* - parametered.

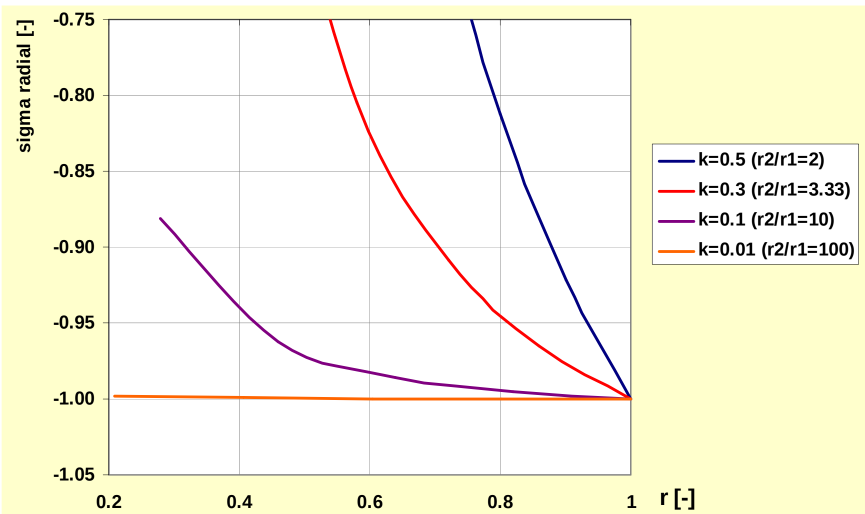

Figure 1. Course of radial stress in relationship with the current radius at different k ratios

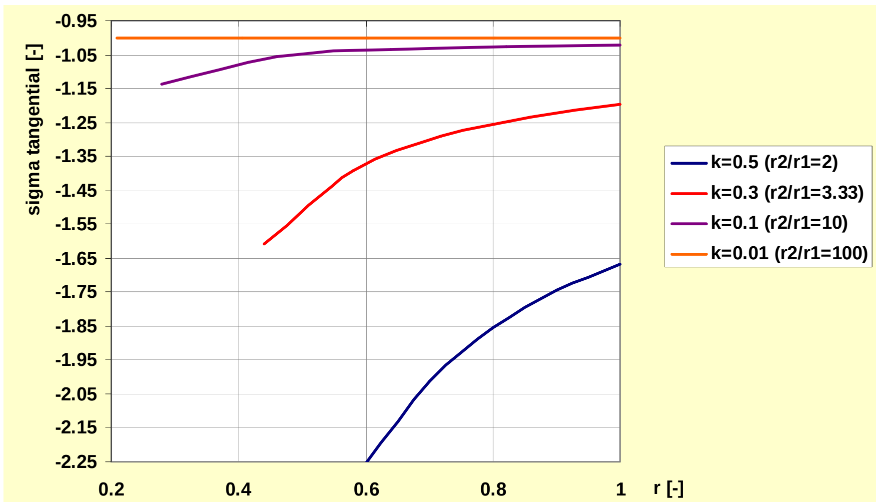

Figure 2. Course of tangential stress in relationship with the current radius at different k ratios

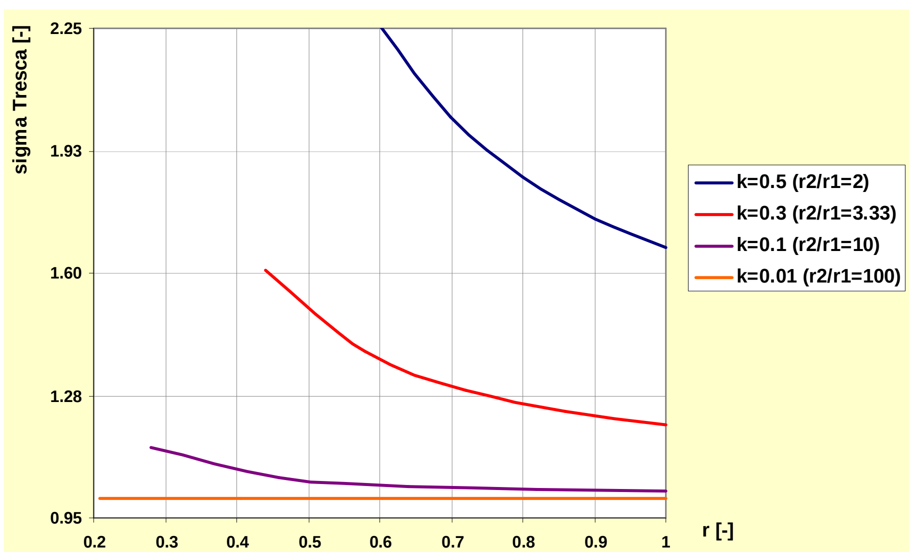

Figure 3. Course of equivalent Tresca stress in relationship with the current radius at different k ratios

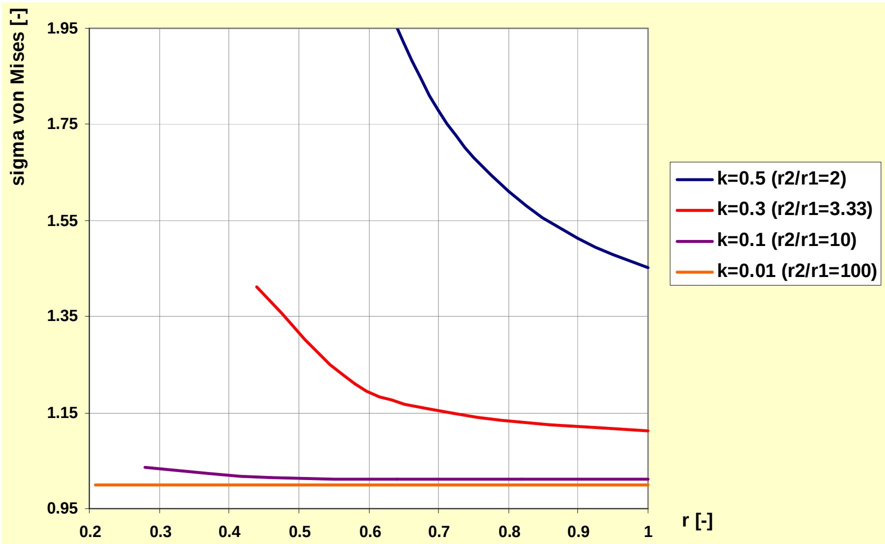

Figure 4. Course of equivalent von Mises stress in relationship with the current radius at different k ratios

3. Discussion and conclusions

Data in table 1 and curves in figures 1…4 allow the following notices and conclusions:

- as k ratio decreases (meaning that the ratio of outer and inner radius grows), the cylinder tends to turn into a solid body. The shape of curves modifies for all stress components. The change in shape shows a decrease of curvature towards an approximate horizontal line, specific to very small values of k
- hence, at high values of radii outer/inner ratio the thick walled cylinders theory becomes inconsistent (relations 18…23 are not valid). Therefore, the solid cylinder may not be approached as a particular case of tube, at least from the standpoint of stress
- one can notice that, as k decreases, stresses tends to homogenize along the radius and become, practically constant, for k<0.01
- an accurate scientific observation states that the curves describing stress variation tend to attain asymptotically ($\sigma_r \rightarrow - p$, $\sigma_t \rightarrow - p$, $\sigma_T \rightarrow p$, $\sigma_M \rightarrow p$) a constant value, equal to the exterior pressure. Applying the limit $r \rightarrow 0$ to thick walled theory does not lead to valid results
- validation of homogenous stress state for the solid shaft is brought by the axial – symmetrical loading theory. This theory states that the compatibility function expressed for the general stress function φ is [2, 3]:

$$\frac{d^4\varphi}{dr^4}+\frac{2d^3\varphi}{rdr^3}-\frac{d^2\varphi}{r^2dr^2}+\frac{d\varphi}{r^3dr}=0 \qquad (24)$$

with the solution:

$$\varphi = A\,log\,r + Br^2\,log\,r + Cr^2 + D\,. \qquad (25)$$

Stress components in cylindrical coordinates are:

$$\sigma_r = \frac{1}{r}\frac{\partial\varphi}{\partial r} = \frac{A}{r^2} + B(1+2\,log\,r)+2C\,, \qquad (26)$$

$$\sigma_t = \frac{\partial^2\varphi}{\partial r^2} = -\frac{A}{r^2} + B(1+2\,log\,r)+2C\,, \qquad (27)$$

$$\tau_{rt} = 0\,, \qquad (28)$$

where A, B and C are constants resulting from boundary conditions. At the inner side of a rotationally symmetrical body (along the axis), constants A and B must be zero (otherwise, for $r \to 0$, stresses become infinite). At the outer side, the boundary condition imposes:

$$\sigma_r = \sigma_t = -p\,. \qquad (29)$$

- therefore, the thick walled cylinders theory shows that the stress tends to homogenize as $k \to 0$, and the axial – symmetrical stress theory validates the homogeneity of stress inside the entire solid cylindrical body, including its axis, where $k = 0$ and $r = 0$. The correct form of relations (18…23) is:

$$\begin{aligned}\sigma_r &= -p\\ \sigma_t &= -p \qquad (30)\\ u &= 0\end{aligned}$$

$$\begin{aligned}\sigma_T &= p\\ \sigma_M &= p\end{aligned} \qquad (31)$$

- from practical point of view, these notices are very useful in estimation of solid shaft loading capacity. Thick walls theory affirms that the maximum pressure taken by a tube is half of the yield strength. For the solid shaft, as equivalent stress (both Tresca and von Mises) in all points is equal to exterior pressure, it results that maximum pressure taken by the shaft is equal to the yield strength of the material. Hence, the solid shaft bears, from the standpoint of interference at least a double loading capacity, compared to the tube equal in outer diameter
- indeed, practical experience shows better strength of solid shafts under exterior pressure. For shaft – hub fits, made of identical materials, the first to plasticize is the hub, which is always hollow. Plastic working is not admitted, in general, by standards. However, the norm DIN 7190 admits plastic regime up to a third of hub's diameter
- considering the above observations, a review of limit strength checking formulae is proposed. The general algorithm should be complete if a specific condition for

the solid shaft was added. So, the elastic working of both shaft and hub is assured if the following conditions are satisfied [4]:

$$p_{max} < p_{c1},\quad p_{max} < p_{c2} \tag{30}$$

where:

$$p_{c1} = \frac{\sigma_{c1}\left[1-\left(\frac{d_1}{d}\right)^2\right]}{2s}, \tag{31}$$

for the hollow shaft

$$p_{c1} = \frac{\sigma_{c1}}{s}, \tag{32}$$

for the solid shaft

$$p_{c2} = \frac{\sigma_{c2}\left[1-\left(\frac{d}{d_2}\right)^2\right]}{2s}, \tag{33}$$

for the hub.

In relations (31…33) s is a safety coefficient applied to limit yield strength σ_c.

Indexes 1 and 2 were assigned to the shaft, respectively to the hub.

Symbols p_{max} and p_c denote maximum pressure induced by the interference, respectively yield pressure of materials.

The nominal dimension of the fit (outer diameter of the shaft and inner diameter of the hub) is d.

References

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