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Nonuniform Exponential Dichotomy of Evolution Operators in Banach Spaces

The aim of this paper is to present notion of the nonuniform exponential dichotomy of evolution operators in terms of nonuniform integral exponential dichotomy.

1. Preliminaries and definitions

Let X be a real or complex Banach space .The norm on X and on the space B(X) of all bounded linear operators on X will be denoted by $\|\cdot\|$.

Definition 1.1 A family $\Phi = \{\Phi(t,s)\}_{t\geq s\geq 0}$ of bounded linear operators on X is called an **evolution operator** if the following properties are satisfied:

O_1) $\Phi(t,t)$**=**I ,the identity operator on X;

O_2) $\Phi(t,s)\Phi(s,t_0) = \Phi(t,t_0)$ for all $t \geq s \geq t_0 \geq 0$;

O_3) for all $x \in X$ and all $t, t_0 \geq 0$, the function $\Phi(t,\cdot)x$ is continuous on $[0,t]$ and the function $\Phi(\cdot,t_0)x$ is continuous on $[t_0,\infty)$**;**

O_4) there exist $M \geq 1, \omega > 0$ such that

$$\|\Phi(t,t_0)\| \leq Me^{\omega(t-t_0)} \tag{1.1}$$

for all $t \geq t_0 \geq 0$.

Remark 1.2 Statement **O_4)** is equivalent with the following :
O_4') there exist a nondecreasing function *f:* $\mathbf{R}_+ \rightarrow \mathbf{R}_+^*$ such that

$$\|\Phi(t,t_0)\| \leq f(t-t_0) \tag{1.2}$$

for all $t \geq t_0 \geq 0$ and all $x \in X$.

Definition 1.3 An evolution operator $\Phi = \{\Phi(t,s)\}_{t\geq s\geq 0}$ is called
(i) nonuniformly stable if there exist a function N *:* $\mathbf{R}_+ \rightarrow \mathbf{R}_+^*$ such that

$$\|\Phi(t,t_0)x\| \le N(t_0)\|x\| \tag{1.3}$$

for all $t \ge t_0 \ge 0$ and all $x \in X$.

(ii) nonuniformly exponentially stable if there exists a function $N : \mathbf{R}_+ \to \mathbf{R}_+^*$ and a constant $\upsilon > 0$ such that

$$\|\Phi(t,t_0)x\| \le N(t_0)e^{-\upsilon(t-t_0)}\|x\| \tag{1.4}$$

for all $t \ge t_0 \ge 0$ and all $x \in X$.

Definition 2.1 is equivalently with

<u>Proposition 1.4</u>(i) if there exists a function $N : \mathbf{R}_+ \to \mathbf{R}_+^*$ such that

$$\|\Phi(t,t_0)x\| \le N(t_0)\|\Phi(s,t_0)x\|$$

for all $t \ge s \ge t_0 \ge 0$ and all $x \in X$ then evolution operator Φ is nonuniformly stable.

(ii) if there exists a function $N : \mathbf{R}_+ \to \mathbf{R}_+^*$ and a constant $\upsilon > 0$ such that

$$\|\Phi(t,t_0)x\| \le N(t_0)e^{-\upsilon(t-s)}\|\Phi(s,t_0)x\|$$

for all $t \ge s \ge t_0 \ge 0$ and all $x \in X$ then evolution operator Φ is nonuniformly exponentially stable .

<u>Definition 1.5</u> The evolution operator Φ is said to be **strongly measurable** if for every $(t_0, x) \in \mathbf{R}_+ \times X$ the function $\|\Phi(\cdot,t_0)x\|$ is measurable**.**

<u>Definition 1.6</u> The strongly measurable evolution operator Φ is said to be

(i) nonuniformly integral stable if there exists a function $M : \mathbf{R}_+ \to \mathbf{R}_+^*$ such that

$$\int_{t_0}^{t} \|\Phi(s,t_0)x\| ds \le M(t_0)\|\Phi(s,t_0)x\| \tag{1.5}$$

for all $t \ge s \ge t_0 \ge 0$ and all $x \in X$;

(ii)nonuniformly integral exponentially stable if there exists a function $M : \mathbf{R}_+ \to \mathbf{R}_+^*$ and a constant $\alpha > 0$ such that

$$\int_{t_0}^{t} e^{\alpha\tau}\|\Phi(\tau,t_0)x\| d\tau \le M(t_0)\|\Phi(s,t_0)x\| \tag{1.6}$$

for all $t \ge s \ge t_0 \ge 0$ and all $x \in X$.

<u>Theorem 1.7</u> A strongly measurable evolution operator Φ is nonuniformly stable if and only if Φ is nonuniformly integral stable.

Proof. *Necessity* .It is a simple verification.

Sufficiency . I. Consider $t_0 \geq 0$, $t \geq t_0 + 1$ and $x \in X$ .We denote by

$$K = \int_0^1 \frac{du}{f(u)}$$

where function f is given as in relation (1.2) .Following relations hold

$$\int_0^1 \frac{\|\Phi(t,t_0)x\|}{f(u)} du = \int_{t-1}^{t} \frac{\|\Phi(t,t_0)x\|}{f(t-s)} ds \leq \int_{t-1}^{t} \|\Phi(s,t_0)x\| ds \leq$$

$$\leq \int_{t_0}^{t} \|\Phi(s,t_0)x\| ds \leq M(t_0)\|x\| .$$

Hence

$$\|\Phi(t,t_0)x\| \leq K^{-1} M(t_0)\|x\|$$

II. If $t_0 \leq t \leq t_0 + 1$ and $x \in X$, we have

$$\|\Phi(t,t_0)x\| \leq f(1)\|x\|$$

If we consider the function $N : \mathbf{R}_+ \to \mathbf{R}_+^*$ defined as

$$N(t_0) = K^{-1} M(t_0) + f(1)$$

we obtain that the evolution operator Φ is nonuniformly stable.

Theorem 1.8 A strongly measurable evolution operator Φ is nonuniformly exponentially stable if and only if Φ is nonuniformly integral exponentially stable.

Proof. *Necessity .*As the evolution operator Φ is nonuniformly exponentially stable there exists function $N : \mathbf{R}_+ \to \mathbf{R}_+^*$ and a constant $\upsilon > 0$ such that

$$\|\Phi(t,s)x\| \leq N(s) e^{-\upsilon(t-s)} \|x\|$$

for all $t \geq s \geq 0$ and all $x \in X$. We consider $\alpha \in (0,\upsilon)$.Following relations hold

$$\int_{t_0}^{t} e^{\alpha\tau} \|\Phi(\tau,t_0)x\| d\tau \leq N(t_0) \int_{t_0}^{t} e^{\alpha\tau} e^{-\upsilon(\tau-t_0)} \|x\| d\tau = M(t_0)\|\Phi(s,t_0)x\|$$

where we have denoted $M(t_0) = \dfrac{N(t_0) e^{\alpha t_0}}{\upsilon - \alpha}$.

Sufficiency. We denote by Φ_λ the mapping $\Phi_\lambda : \mathbf{R}_+^2 \to X$ defined by

$$\Phi_\lambda(t,t_0) = e^{-\lambda(t-t_0)} \Phi(t,t_0).$$

For a constant $\alpha > 0$ consider $\Phi_{-\lambda}(t.t_0)$.Let us denote $\beta = \alpha + \omega$, where ω is given as in Definition 1.1.

If $t \in [t_0, t_0 + 1)$ we have

$$\|\Phi_{-\alpha}(t,t_0)x\| \le Me^{\beta}\|x\|$$

and if $t \in [t_0 + 1, \infty)$ the following relations hold

$$\|\Phi_{-\alpha}(t,t_0)x\| \int_0^1 \frac{1}{M} e^{-\beta\tau} \le \int_{t_0}^{t} \frac{1}{M} e^{-\beta(t-\tau)} \|\Phi_{-\alpha}(t,t_0)x\| d\tau \le M(t_0)\|x\|.$$

We obtain

$$\|\Phi_{-\alpha}(t,t_0)x\| \le \tilde{M}(t_0)\|x\|$$

where we have denoted

$$\tilde{M}(t_0) = Me^{\beta} + M(t_0)\frac{M\beta}{1-e^{-\beta}}.$$

It follows that

$$\|\Phi(t,t_0)x\| \le \tilde{M}(t_0) e^{-\alpha(t-t_0)}\|x\|$$

for $t \ge t_0 \ge 0$ and all $x \in X$ .Hence Φ is nonuniformly exponentially stable.

<u>Definition 1.9</u> The strongly measurable evolution operator $\Phi = \{\Phi(t,s)\}_{t \ge s \ge 0}$ is said to be

(i) nonuniformly instable if there exists a function $N : \mathbf{R}_+ \to \mathbf{R}_+^*$ such that

$$\|\Phi(s,t_0)x\| \le N(t_0)\|\Phi(t,t_0)x\|, \qquad (1.7)$$

for all $t \ge s \ge t_0 \ge 0$ and all $x \in X$.

(ii)nonuniformly exponentially instable if there exists a function $N : \mathbf{R}_+ \to \mathbf{R}_+^*$ and $\upsilon > 0$ such that

$$e^{\upsilon(t-s)}\|\Phi(s,t_0)x_0\| \le N(t_0)\|\Phi(t,t_0)x_0\| \qquad (1.8)$$

for all $t \ge s \ge t_0 \ge 0$ and all $x_0 \in X$.

Definition 1.9 is equivalent with

<u>Proposition 1.10</u> (i) An evolution operator Φ is nonuniformly instable if there exists a function $N : \mathbf{R}_+ \to \mathbf{R}_+^*$ such that

$$N(t_0)\|\Phi(t,t_0)x\| \ge \|x\|,$$

for all $t \ge t_0 \ge 0$ and all $x \in X$.

(ii) An evolution operator Φ is nonuniformly exponentially instable if there exists a function $N : \mathbf{R}_+ \to \mathbf{R}_+^*$ and $\upsilon > 0$ such that

$$N(t_0)\|\Phi(t,t_0)x\| \ge e^{\upsilon(t-t_0)}\|x\|$$

for all $t \ge t_0 \ge 0$ and all $x_0 \in X$.

Proof. Is immediate.

In next we consider a particular class of evolution operators introduced by.

Definition 1.11 The strongly measurable evolution operator Φ is said to be

(i) nonuniformly integral instable if there exist a function $M : \mathbf{R}_+ \to \mathbf{R}_+^*$ such that

$$\int_{t_0}^{t} \|\Phi(s,t_0)x\| ds \le M(t_0) \|\Phi(t,t_0)x_0\| \tag{1.9}$$

for all $t \ge t_0 \ge 0$ and all $x \in X$;

(ii)nonuniformly integral exponentially instable if there exist a function $M : \mathbf{R}_+ \to \mathbf{R}_+^*$ and a constant $\beta > 0$ such that

$$\int_{t_0}^{t} e^{-\beta(t-\tau)} \|\Phi(\tau,t_0)x\| d\tau \le M(t_0) \|\Phi(t,t_0)x_0\| \tag{1.10}$$

for all $t \ge t_0 \ge 0$ and all $x \in X$.

Theorem 1.12 Let Φ be a strongly measurable evolution operator on X . Then Φ is nonuniformly instable if and only if it is uniformly integral instable.

Proof .. It is similar with proof of the Theorem 1.7

Theorem 1.13 Let $\Phi = \{\Phi(t,s)\}_{t \ge s \ge 0}$ be a strongly measurable evolution operator . Then Φ is nonuniformly exponentially instable if and only if it is nonuniformly integral exponentially instable.

Proof. It is similar with proof of the Theorem 1.8

2. Nonuniform exponential dichotomy of evolution operators

Let $\Phi = \{\Phi(t,s)\}_{t \ge s \ge 0}$ be an evolution operator on the Banach space X . The set of all mappings from V into itself will be denoted by F(X).

Definition 2.1 An application P : $\mathbf{R}_+ \to$ F(X) is said to be a **projection family** if $P(t)^2 = P(t)$, for all $t \in \mathbf{R}_+$.

Definition 2.2 Two projection families P_1 and P_2 are said to be **compatible with the evolution operator Φ** if

(cd₁) $P_1(t) + P_2(t) = I, \forall t \ge 0$ (I denotes the identity operator on X);

(cd₂) $P_1(t)P_2(t) = P_2(t)P_1(t) = 0, \forall t \ge 0$ **;**

(cd₃) $\Phi(t,t_0)P_k(t_0) = P_k(t)\Phi(t,t_0), \forall t \ge t_0 \ge 0, k \in \{1,2\}$**.**
In what follows we will denote

$$\Phi_k(t,t_0) = \Phi(t,t_0)P_k(t_0) = P_k(t)\Phi(t,t_0), \forall t \ge t_0 \ge 0, k \in \{1,2\} \quad (2.1)$$

Remark 2.3 If the projection families P_1 and P_2 are compatible with the evolution operator Φ then Φ_1 and Φ_2 are evolution operators on X , which is provided by the following relations

$$\Phi_k(t,s)\Phi_k(s,t_0) = \Phi(t,s)P_k(s)\Phi(s,t_0)P_k(t_0) = P_k(t)\Phi(t,t_0)P_k(t_0) = \Phi_k(t,t_0)$$

for all $t \ge s \ge t_0 \ge 0$, $k \in \{1,2\}$.

Definition 2.4 An evolution operator Φ=$\{\Phi(t,s)\}_{t \ge s \ge 0}$ is said to be **nonuniformly dichotomic** if there exist functions N_1, N_2 :**R₊** $\to$ **R₊***and two projection families P_1 and P_2 are compatible with Φ such that

(ud₁) $\|\Phi_1(t,t_0)x\| \le N_1(t_0)\|\Phi_1(s,t_0)x\|$ **;**

(ud₂) $\|\Phi_2(s,t_0)x\| \le N_2(t_0)\|\Phi_2(t,t_0)x\|$**;**

for all $t \ge s \ge t_0 \ge 0$ and $x \in X$.

Definition 2.5 An evolution operator Φ= $\{\Phi(t,s)\}_{t \ge s \ge 0}$ is said to be **nonuniformly exponentially dichotomic** if there exist functions N_1, N_2:**R₊** $\to$ **R₊***, $\upsilon_1, \upsilon_2 > 0$ and two projection families P_1 and P_2 are compatible with Φ such that

(ued₁) $e^{\upsilon_1(t-s)}\|\Phi_1(t,t_0)x\| \le N_1(t_0)\|\Phi_1(s,t_0)x\|$ **;**

(ued₂) $e^{\upsilon_2(t-s)}\|\Phi_2(s,t_0)x\| \le N_2(t_0)\|\Phi_2(t,t_0)x\|$ **;**

for all $t \ge s \ge t_0 \ge 0$ and $x \in X$.

Remark 2.6 We can suppose in the previous definition that $N_1 = N_2 = N$ and $\upsilon_1 = \upsilon_2 = \upsilon$ because otherwise we consider $N = \max\{N_1, N_2\}$ and $\upsilon = \max\{\upsilon_1, \upsilon_2\}$.

Definition 2.7 An evolution operator Φ=$\{\Phi(t,s)\}_{t \ge s \ge 0}$ is called **nonuniformly integral dichotomic** if there exist functions M_1, M_2 : **R₊** $\to$ **R₊*** such that
(i)

$$\int_{t_0}^{t} \|\Phi_1(\tau,t_0)x\| d\tau \le M_1(t_0)\|\Phi_1(s,t_0)x\| \quad (2.2)$$

(ii)

$$\int_{t_0}^{t} \left\|\Phi_2(\tau,t_0)x\right\| d\tau \le M_2(t_0)\left\|\Phi_2(t,t_0)x\right\| \quad (2.3)$$

for all $t \ge s \ge t_0 \ge 0$ and $x \in X$.

Theorem 2.8 Let Φ be a strongly measurable evolution operator on X . Then Φ is nonuniformly dichotomic if and only if it is nonuniformly integral dichotomic.
Proof. It is result from Theorem 1.7 and Theorem 1.12.

Definition 2.9 An evolution operator Φ= $\{\Phi(t,s)\}_{t\ge s\ge 0}$ is called **nonuniformly integral exponentially dichotomic** if there exists functions $M_1, M_2 : \mathbf{R}_+ \to \mathbf{R}_+^*$ such that
(i)

$$\int_{t_0}^{t} e^{\alpha(\tau-t_0)}\left\|\Phi_1(\tau,t_0)x\right\| d\tau \le M_1(t_0)\left\|\Phi_1(s,t_0)x\right\| \quad (2.4)$$

(ii)

$$\int_{t_0}^{t} e^{\alpha(t-\tau)}\left\|\Phi_2(\tau,t_0)x\right\| d\tau \le M_2(t_0)\left\|\Phi_2(t,t_0)x\right\| \quad (2.5)$$

for all $t \ge s \ge t_0 \ge 0$ and $x \in X$.

Theorem 2.10 Let Φ be a strongly measurable evolution operator on X Then Φ is uniformly exponentially dichotomic if and only if Φ is uniformly integral exponentially dichotomic.
Proof . It is result from Theorem 1.8 and Theorem 1.13.

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