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Nonuniform Exponential Stability and Instability of Evolution Operators in Banach Spaces

In this paper is presenting a parallel between nonuniform exponential stability and nonuniform exponential instability of evolution operators in Banach spaces, beginning to present the concept of the evolution operator with nonuniform exponential decay, respectively growth, next with the concept of the nonuniform stability, respectively instability, nonuniform exponential stability, respectively instability, nonuniform integrable stability, respectively instability and relationship between this concepts.

1. Introduction

Let X be a real or complex Banach space .The norm on X and on the space B(X) of all bounded linear operators on X will be denoted by $\|\cdot\|$.

Definition 1.1 A family $\Phi = \{\Phi(t,s)\}_{t \geq s \geq 0}$ of bounded linear operators on X is called an **evolution operator** if the following properties are satisfied:

O₁) $\Phi(t,t)$ **=** I ,the identity operator on X;

O₂) $\Phi(t,s)\Phi(s,t_0) = \Phi(t,t_0)$ for all $t \geq s \geq t_0 \geq 0$;

O₃) for all $x \in X$ and all $t, t_0 \geq 0$, the function $\Phi(t,\cdot)x$ is continuous on $[0,t]$ and the function $\Phi(\cdot,t_0)x$ is continuous on $[t_0,\infty)$;

O₄) there exist $M \geq 1, \omega > 0$ such that

$$\|\Phi(t,t_0)\| \leq Me^{\omega(t-t_0)} \tag{1.1}$$

for all $t \geq t_0 \geq 0$

A particular class of evolution operators is introduced by

Definition 1.2 An evolution operator $\Phi = \{\Phi(t,s)\}_{t \geq s \geq 0}$ is called

(i) with nonuniform exponential decay if there exist a function M : $\mathbf{R}_+ \longrightarrow \mathbf{R}_+^*$ and $\omega > 0$ such that

$$\|\Phi(s,t_0)x\| \le M(t_0)e^{\omega(t-s)}\|\Phi(t,t_0)x\|,$$

for all $t \ge s \ge t_0 \ge 0$ and all $x \in X$;

(ii)with nonuniform exponential growth if there exist a function M : $\mathbf{R}_+ \longrightarrow \mathbf{R}_+^*$ and $\omega > 0$ such that

$$\|\Phi(t,t_0)x\| \le M(t_0)e^{\omega(t-s)}\|\Phi(s,t_0)x\|,$$

for all $t \ge s \ge t_0 \ge 0$ and all $x \in X$.

Remark 1.3 Statement **O_4)** is equivalent with the following :
O_4') there exist a nondecreasing function *f:* $\mathbf{R}_+ \longrightarrow \mathbf{R}_+^*$ such that

$$\|\Phi(t,t_0)\| \le f(t-t_0) \tag{1.2}$$

for all $t \ge t_0 \ge 0$ and all $x \in X$.

2. Nonuniform exponential stability of evolution operators in Banach spaces

Definition 2.1 An evolution operator $\Phi = \{\Phi(t,s)\}_{t \ge s \ge 0}$ is called

(i) nonuniformly stable if there exist a function N : $\mathbf{R}_+ \longrightarrow \mathbf{R}_+^*$ such that

$$\|\Phi(t,t_0)x\| \le N(t_0)\|x\| \tag{2.1}$$

for all $t \ge t_0 \ge 0$ and all $x \in X$.

(ii) nonuniformly exponentially stable if there exists a function N : $\mathbf{R}_+ \longrightarrow \mathbf{R}_+^*$ and a constant $\upsilon > 0$ such that

$$\|\Phi(t,t_0)x\| \le N(t_0)e^{-\upsilon(t-t_0)}\|x\| \tag{2.2}$$

for all $t \ge t_0 \ge 0$ and all $x \in X$.

Definition 2.1 is equivalently with
Proposition 2.2 **(i)** if there exists a function N : $\mathbf{R}_+ \longrightarrow \mathbf{R}_+^*$ such that

$$\|\Phi(t,t_0)x\| \le N(t_0)\|\Phi(s,t_0)x\| \tag{2.3}$$

for all $t \ge s \ge t_0 \ge 0$ and all $x \in X$ then evolution operator Φ is nonuniformly stable.

(ii) if there exists a function N : $\mathbf{R}_+ \longrightarrow \mathbf{R}_+^*$ and a constant $\upsilon > 0$ such that

$$\|\Phi(t,t_0)x\| \le N(t_0)e^{-\upsilon(t-s)}\|\Phi(s,t_0)x\| \tag{2.4}$$

for all $t \geq s \geq t_0 \geq 0$ and all $x \in X$ then evolution operator Φ is nonuniformly exponentially stable .

Definition 2.3 The evolution operator Φ is said to be **strongly measurable** if for every $(t_0, x) \in$ **R₊**$\times X$ the function $\|\Phi(\cdot, t_0)x\|$ is measurable**.**

Definition 2.4 Let Φ be a strongly measurable evolution operator. Φ is said to be **nonuniformly integral stable** if there exists a function $M : \mathbf{R}_+ \to \mathbf{R}_+^*$ such that

$$\int_{t_0}^{t} \|\Phi(s,t_0)x\| ds \leq M(t_0)\|\Phi(s,t_0)x\| \qquad (2.5)$$

for all $t \geq s \geq t_0 \geq 0$ and all $x \in X$.

Theorem 2.5 A strongly measurable evolution operator Φ is nonuniformly stable if and only if Φ is nonuniformly integral stable.

Proof. *Necessity .*It is a simple verification.

Sufficiemcy . I. Consider $t_0 \geq 0$, $t \geq t_0 + 1$ and $x \in X$ .We denote by

$$K = \int_0^1 \frac{du}{f(u)}$$

where function f is given as in relation (1.2) .Following relations hold

$$\int_0^1 \frac{\|\Phi(t,t_0)x\|}{f(u)} du = \int_{t-1}^{t} \frac{\|\Phi(t,t_0)x\|}{f(t-s)} ds \leq \int_{t-1}^{t} \|\Phi(s,t_0)x\| ds \leq$$

$$\leq \int_{t_0}^{t} \|\Phi(s,t_0)x\| ds \leq M(t_0)\|x\| .$$

Hence

$$\|\Phi(t,t_0)x\| \leq K^{-1} M(t_0)\|x\|$$

II. If $t_0 \leq t \leq t_0 + 1$ and $x \in X$, we have

$$\|\Phi(t,t_0)x\| \leq f(1)\|x\|$$

If we consider the function $N : \mathbf{R}_+ \to \mathbf{R}_+^*$ defined as

$$N(t_0) = K^{-1} M(t_0) + f(1)$$

we obtain that the evolution operator Φ is nonuniformly stable.

<u>Definition 2.6</u> Let Φ be a strongly measurable evolution operator. Φ is said to be **nonuniformly integral exponentially stable** if there exists a function M : $\mathbf{R}_+ \rightarrow \mathbf{R}_+^*$ and a constant $\alpha > 0$ such that

$$\int_{t_0}^{t} e^{\alpha\tau} \|\Phi(\tau, t_0)x\| d\tau \le M(t_0) \|\Phi(s, t_0)x\| \qquad (2.6)$$

for all $t \ge s \ge t_0 \ge 0$ and all $x \in X$.

<u>Theorem 2.7</u> A strongly measurable evolution operator Φ is nonuniformly exponentially stable if and only if Φ is nonuniformly integral exponentially stable.

Proof. *Necessity* .As the evolution operator Φ is nonuniformly exponentially stable there exists function N : $\mathbf{R}_+ \rightarrow \mathbf{R}_+^*$ and a constant $\upsilon > 0$ such that

$$\|\Phi(t, s)x\| \le N(s) e^{-\upsilon(t-s)} \|x\|$$

for all $t \ge s \ge 0$ and all $x \in X$. We consider $\alpha \in (0, \upsilon)$.Following relations hold

$$\int_{t_0}^{t} e^{\alpha\tau} \|\Phi(\tau, t_0)x\| d\tau \le N(t_0) \int_{t_0}^{t} e^{\alpha\tau} e^{-\upsilon(\tau - t_0)} \|x\| d\tau = M(t_0) \|\Phi(s, t_0)x\|$$

where we have denoted $M(t_0) = \dfrac{N(t_0) e^{\alpha t_0}}{\upsilon - \alpha}$.

Sufficiency. We denote by Φ_λ the mapping Φ_λ :$\mathbf{R}_+^{\mathbf{2}} \rightarrow X$ defined by

$$\Phi_\lambda(t, t_0) = e^{-\lambda(t - t_0)} \Phi(t, t_0).$$

For a constant $\alpha > 0$ consider $\Phi_{-\lambda}(t.t_0)$.Let us denote $\beta = \alpha + \omega$, where ω is given as in Definition 1.1.

If $t \in [t_0, t_0 + 1)$ we have

$$\|\Phi_{-\alpha}(t, t_0)x\| \le M e^{\beta} \|x\|$$

and if $t \in [t_0 + 1, \infty)$ the following relations hold

$$\|\Phi_{-\alpha}(t, t_0)x\| \int_0^1 \frac{1}{M} e^{-\beta\tau} \le \int_{t_0}^{t} \frac{1}{M} e^{-\beta(t-\tau)} \|\Phi_{-\alpha}(t, t_0)x\| d\tau \le M(t_0) \|x\|.$$

We obtain

$$\|\Phi_{-\alpha}(t, t_0)x\| \le \tilde{M}(t_0) \|x\|$$

where we have denoted

$$\tilde{M}(t_0) = M e^{\beta} + M(t_0) \frac{M\beta}{1 - e^{-\beta}}.$$

It follows that

$$\|\Phi(t,t_0)x\| \le M(t_0) e^{-\tilde{\alpha}(t-t_0)} \|x\|$$

for $t \ge t_0 \ge 0$ and all $x \in X$ .Hence Φ is nonuniformly exponentially stable.

3. Nonuniform exponential instability of evolution operators in banach spaces

Definition 3.1 The evolution operator $\Phi = \{\Phi(t,s)\}_{t \ge s \ge 0}$ is said to be

(i) nonuniformly instable if there exists a function $N : \mathbf{R}_+ \to \mathbf{R}_+^*$ such that

$$\|\Phi(s,t_0)x\| \le N(t_0)\|\Phi(t,t_0)x\|, \tag{3.1}$$

for all $t \ge s \ge t_0 \ge 0$ and all $x \in X$.

(ii) nonuniformly exponentially instable if there exists a function $N : \mathbf{R}_+ \to \mathbf{R}_+^*$ and $\upsilon > 0$ such that

$$e^{\upsilon(t-s)}\|\Phi(s,t_0)x_0\| \le N(t_0)\|\Phi(t,t_0)x_0\| \tag{3.2}$$

for all $t \ge s \ge t_0 \ge 0$ and all $x_0 \in X$.

Definition 3.1 is equivalent with

Proposition 3.2 **(i)** An evolution operator Φ is nonuniformly instable if there exists a function $N : \mathbf{R}_+ \to \mathbf{R}_+^*$ such that

$$N(t_0)\|\Phi(t,t_0)x\| \ge \|x\|, \tag{3.3}$$

for all $t \ge t_0 \ge 0$ and all $x \in X$.

(ii) An evolution operator Φ is nonuniformly exponentially instable if there exists a function $N : \mathbf{R}_+ \to \mathbf{R}_+^*$ and $\upsilon > 0$ such that

$$N(t_0)\|\Phi(t,t_0)x\| \ge e^{\upsilon(t-t_0)}\|x\| \tag{3.4}$$

for all $t \ge t_0 \ge 0$ and all $x_0 \in X$.

Proof. Is immediate.

In next we consider a particular class of evolution operators introduced by.

Definition 3.3 Let Φ be a strongly measurable evolution operator. Φ is said to be **nonuniformly integral_instable**_if there exist a function $M : \mathbf{R}_+ \to \mathbf{R}_+^*$ such that

$$\int_{t_0}^{t} \|\Phi(s,t_0)x\| ds \le M(t_0)\|\Phi(t,t_0)x_0\| \tag{3.5}$$

for all $t \geq t_0 \geq 0$ and all $x \in X$.

Theorem 3.4 Let Φ be a strongly measurable evolution operator on X . Then Φ is nonuniformly instable if and only if it is uniformly integral instable.
Proof . It is similar with proof of the Theorem 2.5 .

Definition 3.5 Let Φ be a strongly measurable evolution operator. Φ is said to be **nonuniformly integral exponentially instable** if there exist a function M : $\mathbf{R}_+ \to \mathbf{R}_+^*$ and a constant $\beta > 0$ such that

$$\int_{t_0}^{t} e^{-\beta(t-\tau)} \left\| \Phi(\tau, t_0) x \right\| d\tau \leq M(t_0) \left\| \Phi(t, t_0) x_0 \right\| \tag{3.6}$$

for all $t \geq t_0 \geq 0$ and all $x \in X$.

Theorem 3.6 Let $\Phi = \{\Phi(t,s)\}_{t \geq s \geq 0}$ be a strongly measurable evolution operator. Then Φ is nonuniformly exponentially instable if and only if it is nonuniformly integral exponentially instable.
Proof. It is similar with proof of the Theorem 2.7

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