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Numerical Model for the Study of Electromagnetic Processes in non-Stationary Regime

The non-steady (non-stationary) state of many electromagnetic processes encountered in the industrial practice may be considered as quasi-stationary, in the sense that in all conducting parts one may neglect the capacitance current compared to the line current. The most frequently used model for such a process is the mathematical model of electromagnetic potentials, which consists mainly of the equation:

$$\nabla \times \left(\nabla \times \overline{A}\right) + \mu\sigma \frac{\partial \overline{A}}{ct} = \mu \overline{J}_s \qquad (1)$$

Where: $\overline{A}$ is the vector magnetic potential, μ, σ are the magnetic permeability and the electric conductivity respectively, and $\overline{J}_s$ is the density of the electric current furnished by the power supply sources (power units) of the device in which the studied process intervenes. Knowing the quantities $\mu, \sigma, \overline{J}_s$ and adding to equation (1) the sampling condition and the border conditions, we obtain through its integration the quantity $\overline{A}$, which enables us to determine all the parameters describing the respective process. However, the direct integration of the differential equation (1) is possible only in several specific cases, which imposes the use of numerical methods. In principle through these methods equation (1) is turned into a system of equations where the unknown terms are the values of the quantity $\overline{A}$ in certain points within the range of occurrence of the studied process. The paper uses the method of the finite elements and we determine the matrix of the unknown coefficients as well as the column vector of the constant terms

1. Introduction

In principle through these methods we transform equation (1) in a system of equations in which the unknown terms are the values of the quantity $\overline{A}$ in certain

points of the range within which the studied process takes place. In this paper the numerical model is obtained with the method of finite elements, which requires firstly that the range within which the studied process takes place should be divided into disjoint segments, called finite elements. These are interconnected among themselves through certain points called nodes.

2. Deducing the numerical model

For simplification we introduce the notation:

$$\overline{a} = \nabla \times \nabla \times \overline{A} \tag{2}$$

Consequently equation (1) becomes:

$$\mu\sigma\frac{\partial \overline{A}}{\partial t} + \overline{a} = \mu\,\overline{I}_s \tag{3}$$

The method of the finite elements uses four scalar equations, and relation (3) is a vector equation. For this reason we transform (3) in a scalar one, using the method of decomposing into components along the directions of the axes of the chosen system of co-ordinates. In general, the axes of the system of co-ordinates are noted with x, y, z, but, in order to express the matrix condenser, the relations obtained where we will have the notations: x_1, x_2, x_3. Also, the component of any vector quantity along each axis will be noted using the symbol used to designate that vector quantity, an index that is the free index x for the respective axis. Considering that the respective range is 3D, according to the above, equation 3 is transformed into 3 scalar equations:

$$\mu\sigma\frac{\partial A_a}{\partial t} + Q_a = \mu I_{sa}, \ a = \overline{1,3} \tag{4}$$

In equation 4 the unknown terms are the values of the components of the vector magnetic potential $\overline{A}$, along the axes, i.e. A_a which generally varies both in time and space, i.e.:

$$A_a = A_a(t,P), \ a = \overline{1,3}, \tag{5}$$

where t is a random moment, and P represents the co-ordinates of a certain point P from the range where the studied process takes place.

The functions $A_a(t,P)$ usually have a continuous variation in time and space.

The finite elements method belongs to the category of numerical methods of approximation through discretisation. Within such a method, each of the A_α functions is replaced with another one, called approximation of the respective function, noted with $\hat{A}_\alpha$, which exhibits a discrete variation in time and space.

Further on, the space discretisation is realised for each finite element. We consider an arbitrary finite element, with a p number of nodes numbered in a cer-

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tain order 1, 2, 3, ..., p, called local indexation of nodes for the considered finite element. One of its arbitrary nodes is marked with j, and the values of the functions A_α in the respective node will be noted with A_{ja}, $a = \overline{1,3}$, $j = \overline{1,p}$. Obviously these values are constant in space, but they depend on the chosen t moment, i.e.:

$$A_{ja} = A_{ja}(t), \quad a = \overline{1,3}, \quad j = \overline{1,p} \tag{6}$$

The numerical model for the considered finite element, called elemental numerical mode, containing as unknown terms precisely the values A_{ja}, $a = \overline{1,3}$, $j = \overline{1,p}$ so a number of 3p unknown terms, so we shall have 3p equations. Of course this is valid for each concrete moment. In order to obtain the elemental numerical model corresponding to the arbitrary moment, each node is attached a function which for the arbitrary node is noted with N_j.

Thus, considering that P is a random point of the D range of the finite element considered, the N_j functions are of the following form:

$$N_j = N_j(P), \quad j = \overline{1,p} \tag{7}$$

As the $N_j(P)$ functions depend on the shape of the finite element considered, they are called form functions attached to the nodes of the respective finite element. Moreover, the $\hat{A}_a$ approximations for P are chosen of the form:

$$\hat{A}_a = \sum_{j=1}^{p} N_j A_{ja}, \quad a = \overline{1,3}, \quad P \in D \tag{8}$$

It is obvious that all the above remarks on the A_a components remain valid also for the a_a components, i.e. for each a_a component we define an approximation marked with $\hat{a}_a$.

But, taking into consideration relation (2), the $\hat{a}_a$ approximations are no longer defined through relations of the (8) form, and they must be expressed depending on the $\hat{A}_\alpha$ approximation. For this purpose we define the vector approximation of a vector quantity as being also a vector quantity having as components the approximations of the size of the respective vector quantity. So the vector approximations of the quantities $\overline{A}, \overline{a}$, noted with $\hat{\overline{A}}$ and $\hat{\overline{a}}$ respectively have as components the approximations $\hat{A}_a$ and $\hat{a}_a$, $a = 1,3$, respectively

Besides, it is obvious that relation (2) remains valid even if the vector quantities $\overline{a}, \overline{A}$ are replaced with their vector approximations, i.e.:

$$\hat{\overline{u}} = \nabla \times \nabla \times \hat{\overline{A}} \tag{9}$$

It is easily verified that, from the vector point of view, relation (8) is:

$$\overline{\overline{A}} = \sum_{j=1}^{p} N_j \overline{A}_j \, , \tag{10}$$

where $\overline{A}_j$ is the value of the vector quantity $\overline{A}$ in the knot j, having as components the quantities $A_{j\alpha}$, $\alpha = 1,3$.

After replacing relation (10) in relation (9) we obtain:

$$\overline{\overline{a}} = \nabla \times \nabla \times \sum_{j=1}^{p} N_j \overline{A}_j \tag{11}$$

Taking into consideration that in the ratio with the rotor operator the vector quantities $\overline{A}_j$ are constant, relation (11) becomes:

$$\overline{\overline{a}} = \sum_{j=1}^{p} \left[\left(\overline{A}_j \cdot \nabla \right) \cdot \left(\nabla N_j \right) - \left(\Delta N_j \right) \overline{A}_j \right], \tag{12}$$

where Δ is the Laplace operator.

Relation (12) shows that the approximations $\hat{a}_\alpha$, $\alpha = \overline{1,3}$ are the values of the components of the right member of this relation, and in order to determine them we must first express the right member analytically.

First, noting with $\overline{u}_\alpha$, $\alpha = \overline{1,3}$, the unit vectors of axes, the vector quantities $\overline{A}_j$ are expressed analytically through the relation:

$$\overline{A}_j = \sum_{\alpha=1}^{3} A_{j\alpha} \overline{u}_\alpha, \quad j = \overline{1,p} \tag{13}$$

For instance, if we use a Cartesian system of co-ordinates, then the operator with the analytical expressing:

$$\Delta = \sum_{\alpha=1}^{3} \frac{\partial}{\partial x_\alpha} \overline{u}_\alpha \tag{14}$$

For this case in the paper [1] it is shown that the expressions $\hat{u}_\alpha$ are given by the relations:

$$\hat{a}_\alpha = \sum_{j=1}^{p} \left(\sum_{\beta=1}^{3} C_{\alpha\beta}^{j} A_{j\beta} \right), \quad \alpha = \overline{1,3}, \tag{15}$$

where:

$$C_{\alpha\beta}^{j} = \frac{\partial}{\partial x_\alpha} \left(\frac{\partial N_j}{\partial x_\beta} \right) - \delta_{\alpha\beta} \Delta N_j, \tag{16}$$

in which $\delta_{\alpha\beta}$ is the Kroneker's operator.

For obtaining the numerical model it is necessary that the differential mathematical models be transformed into integral models on the D range of the

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finite element considered. For this purpose we use the method of weighed residues in the Galerkin variant, which for each knot i of the finite element leads to an integral model of the form:

$$\mu\sigma\int_D N_i \frac{\partial \hat{A}_a}{\partial t} dD + \int_D N_i \hat{a}_a dD = \mu\int_D N_i J_{Da} dD, \qquad i = \overline{1,p}, \ a = \overline{1,3} \ (17)$$

In relation (17) we replace the approximations $\hat{A}_\alpha$, $\hat{a}_\alpha$ by their expressions given by relations (8), relation (15).

In the mentioned paper it is shown that for the second term of the left member of the relation (17) we obtain:

$$\int_D N_i \hat{a}_a dD = \sum_{j=1}^{p}\left(\sum_{\beta=1}^{3} C_{a\beta}^{ji} A_{j\beta}\right), \quad i = \overline{1,p}, \ a = \overline{1,3} \tag{18}$$

where:

$$C_{a\beta}^{ji} = \int_D N_i C_{a\beta}^{j} dD \tag{19}$$

We also introduce the notation:

$$F_a^j = \int_D N_i J_{sa}, \quad i = \overline{1,p}, \ a = \overline{1,3} \tag{20}$$

For the first term of the left member of relation (17) we arrive at:

$$\int_D N_i \frac{\partial \hat{A}_a}{\partial t} dD = \int_D \left[N_i \frac{\partial}{\partial t}\left(\sum_{j=1}^{p} N_j A_{ja}\right)\right] dD \tag{21}$$

Considering the property of the derivative of a sum and the fact that the functions of the N_j type have only one spatial variation, relation (21) becomes:

$$\int_D N_i \frac{\partial \hat{A}_a}{\partial t} dD = \int_D \left[N_i \sum_{j=1}^{p} N_j \frac{\partial A_{ja}}{\partial t}\right] dD \tag{22}$$

Taking into account the fact that depending on the summing operation, N_j is a constant, we may introduce under the sum and it follows:

$$\int_D N_i \frac{\partial \hat{A}_a}{\partial t} = \int_D \left(\sum_{j=1}^{p} N_i N_j \frac{\partial A_{ja}}{\partial t}\right) dD \tag{23}$$

Using the property of an integral referring to a sum, relation (23) changes into:

$$\int_D N_i \frac{\partial \hat{A}_a}{\partial t} dD = \sum_{j=1}^{p} T_{ij} \frac{\partial}{\partial t}(A_{ij}), \quad i = \overline{1,p}, \ a = \overline{1,3}, \tag{24}$$

where:

$$T_{ij} = \int_D N_i N_j d, \quad i = \overline{1,p}, \ j = \overline{1,p} \tag{25}$$

In relation (17) we replace relations (25), (18), (20) and we get:

$$\mu\sigma\sum_{j=1}^{p}T_{ij}\frac{\partial}{\partial t}(A_{ja})+\sum_{j=1}^{p}(\sum_{\beta=1}^{3}C_{a\beta}^{ji}A_{j\beta})=\mu F_{a}^{i}, i=\overline{1,p}, a=\overline{1,3} \qquad (26)$$

From relations (19) and (16) we obtain:

$$C_{a\beta}^{ji}=\int_{D}Ni\left[\frac{\partial}{\partial x_{a}}\left(\frac{\partial N_{j}}{\partial x_{\beta}}\right)-\delta_{a\beta}DN_{j}\right] \qquad (27)$$

Having chosen the form functions with relations (25), (27) we calculate the coefficients T_{ij}, $C_{a\beta}^{ji}$, $i=\overline{1,p}$, $j=\overline{1,p}$, $a=\overline{1,3}$, $\beta=\overline{1,3}$. Moreover, knowing the components of the current density I_{sa}, $a=\overline{1,3}$ with relation (20) we compute the free terms F_{a}^{i}, $i=\overline{1,p}$, $a=\overline{1,3}$.

At the same time, as shown, through A_{ja}, $a=\overline{1,3}$ we must understand the values of the components of the vector quantity $\overline{A_{j}}$, $j=\overline{1,p}$ at a certain arbitrary moment and using a diagram with finite differences, the partial derivative against t is expressed depending on the values A_{ja} and the values of the same components at one or several prior moments. Considering that the values of the components at the initial moment have been imposed, it means that the values of the components at any moment prior to the considered one are already known.

In conclusion, in relations (26) the only unknown terms are the values of components A_{ja}, $j=\overline{1,p}$, $a=\overline{1,3}$.

With the values p, P, I determined, their real values, noted with p_b, P_b, I_b respectively are calculated from relations (15), (16), (17).

3. The matrix expression

In order to obtain the system of equations, firs tin relations (27) we develop the sum depending on p and we obtain:

$$\sum_{j=1}^{p}\left[\mu\sigma T_{ij}\frac{\partial}{\partial t}\left(A_{ja}\right)+C_{a1}^{ji}A_{j1}+C_{a2}^{ji}A_{j2}+C_{a3}^{ji}A_{j3}\right]=\mu F_{a}^{i} \qquad (28)$$

For implementing the solution of this system of equations on a computer we should express it under matrix form.

For this purpose we note with [A] the matrix of the unknown terms, which will be a column matrix (column vector) with the dimension 3px1

Keeping in mind the above, 3 consecutive rows of the matrix represent the values of components $\overline{A}$ of one node at a time, noted in the sequence: 1, 2, 3, …, p. This means that:

$$[A]=\left[A_{11}A_{12}A_{13}A_{21}A_{22}A_{23}...A_{p1}A_{p2}A_{p3}\right]^{T}, \qquad (29)$$

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where the exponent T indicates the operation of transposition of the matrix.

We introduce the notation:

$$\frac{\partial}{\partial t}[A] = \left[\frac{\partial A_{11}}{\partial t} \ \frac{\partial A_{12}}{\partial t} \ \frac{\partial A_{13}}{\partial t} \ \frac{\partial A_{21}}{\partial t} \ \frac{\partial A_{22}}{\partial t} \ \frac{\partial A_{23}}{\partial t} \ ... \ \frac{\partial A_{p1}}{\partial t} \ \frac{\partial A_{p2}}{\partial t} \ \frac{\partial A_{p3}}{\partial t} \right]^T \qquad (30)$$

With this, matricially, the system of equations is written:

$$\mu\sigma[M]\frac{\partial}{\partial t}[A] + [K][A] = [F], \qquad (31)$$

where [F] is a column matrix with the dimensions 3px1, and its elements are calculated with the relations (20).

It is clear that in [11] and [12] we must have square matrices with the 3px3p and their elements are determined with relations (26) and (27)

Taking into account the above on the model in which the equations of the system are obtained, it follows that for any matrix, the rows are obtained allotting to i the values 1, 2, 3, ..., p, and for each value attributed for i we get 3 consecutive rows obtained for the values $a = 1, 2, 3$.

Consequently, for an arbitrary row noted with γ for the values of i and α are given by the relations:

$$i = round\left(\frac{\gamma}{3}\right) \qquad (32)$$

$$\gamma = \overline{np}$$

$$a = \begin{cases} 3, if \ \gamma \bmod 3 = 0 \\ \gamma \bmod 3, if \ \gamma \bmod 3 \neq 0 \end{cases} \qquad (33)$$

$$\gamma = \overline{1,3p}$$

where $round\left(\frac{\gamma}{3}\right)$ is the function of superior rounding having as result the natural

number equal or immediately superior to the value $\frac{\gamma}{3}$, γ mod 3 is the rest of the integer division of γ by 3.

The term of the γ row of the matrix [F] noted with F_γ, $\gamma = \overline{1,p}$, is calculated with relations (20), where i and α have the values given by relations (32) and (33) respectively.

In the paper [3] it is shown that for an arbitrary column noted with δ of matrix [K] the values of j and β are given by the relations:

$$j = round\left(\frac{\delta}{3}\right) \qquad (34)$$

$$\delta = \overline{1, p}$$

$$\beta = \begin{cases} 3, if \ \delta \ \text{mod} \ 3 = 0 \\ \delta \ if \ 3, dac\u0103 \ \delta \ \text{mod} \ 3 \neq 0 \end{cases}$$ (35)

$$\delta = \overline{1,3}p$$

For the calculus of the elements of matrix [M] we use the relations (25), defining a square matrix with the dimensions p x p, but the matrix [M] has the dimensions 3p x 3p. Consequently each row of the matrix [M] some elements have the values given by relations (25) and the other are null. Similarly to the case of the other two matrices, for an arbitrary row of matrix [M], noted with γ, the index i of relations (25) is calculated with relation (32).

It is easy to observe that the values of δ for which we obtain k = the relation gives j:

$$\delta = 3k + a - 3$$ (36)

From relation (36) it follows:

$$k = \frac{\delta + 3 - a}{3}$$ (37)

4. Conclusion

The numerical model presented has the advantage of being a rather simple model and can be implemented on an electronical computer with any programming language.

References

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