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Perron Conditions for Stability of Linear Skew-Product Semiflows in Banach Spaces

In this paper we give necessary and sufficient conditions for uniform exponential stability of linear skew-product semiflows in Banach spaces. We give theorems of characterization for uniform exponential stability of linear skew-product semiflows in terms of boundedness of some operators acting on $C_{00}(\mathbb{R}_+, X), C_b(\mathbb{R}_+, X), C_c(\mathbb{R}_+, X)$ *and* $L^p(\mathbb{R}_+, X)$*, respectively.*

1. Linear skew-product semiflow

Let X be a Banach space, let (Θ, d) be a metric space and let $E = X \times \Theta$. We shall denote by $B(X)$ the Banach algebra of all bounded linear operators from X into itself. Throughout the paper, de norm on X and on $B(X)$ will be denoted by $\|\cdot\|$.

Definition 1.1 A continuous mapping $\sigma : \Theta \times \mathbb{R}_+ \to \Theta$ is said to be a **semiflow** on Θ, if it has the following properties:

(i) $\sigma(\theta, 0) = \theta$, $\forall \theta \in \Theta$

(ii) $\sigma(\theta, s+t) = \sigma(\sigma(\theta, s), t)$ for $\forall (\theta, s, t) \in \Theta \times \mathbb{R}_+^2$

Definition 1.2 A pair $\pi = (\Phi, \sigma)$ is called **linear skew-product semiflow** on $E = X \times \Theta$ if σ is a semiflow on Θ and $\Phi : \Theta \times \mathbb{R}_+ \to B(X)$ satisfies the following conditions:

(i) $\Phi(\theta, 0) = I$, the identity operator on X, $\forall \theta \in \Theta$

(ii) $\Phi(\theta, t+s) = \Phi(\sigma(\theta, s), t)\Phi(\theta, s)$ for all $(\theta, t, s) \in \Theta \times \mathbb{R}_+^2$ (the cocycle identity)

(iii) $\exists M \geq 1$ and $\omega > 0$ such that $\|\Phi(\theta, t)\| \leq Me^{\omega t}$ for $\forall (\theta, t) \in \Theta \times \mathbb{R}_+$

If, in addition,
(iv) for every $(x,\theta)\in E$ the mapping $t\to\Phi(\theta,t)x$ is continuous then π is called a strongly continuous linear skew-product semiflow.

Remark 1.1 Statement (iii) is equivalent with the following
(iii)′ there exist a nondecreasing function $f:\mathrm{R}_+\to\mathrm{R}_+^*$ such that $\left\|\Phi(\theta,t)\right\|\leq f(t)$ for all $(\theta,t)\in\theta\times\mathrm{R}_+$.

Example 1.1
Let X be a Banach space. We consider $C(\mathrm{R}_+,\mathrm{R})$ the space of all continuous function with the topology of uniform convergence on compact subsets on R_+. This space is metrizable with the metric
$d(x,y)=\sum_{n=1}^{\infty}\frac{1}{2^n}\cdot\frac{d_n(x,y)}{1+d_n(x,y)}$ where $d_n(x,y)=\sup_{t\in[0,n]}\left|x(t)-y(t)\right|$.
On the Banach space X, we consider the nonautonomous differential equation

$$\dot{x}(t)=a(t)x(t)\ ,\ t\geq 0$$

where $a:\mathrm{R}_+\to\mathrm{R}_+$ be a uniformly continuous function such that there exists $\alpha:=\lim_{t\to\infty}a(t)<\infty$. If we denote by $a_s(t)=a(t+s)$ and by $\Theta=\overline{\{a_s:s\in\mathrm{R}_+\}}$ then $\sigma(\theta,t)(s):=\theta(t+s)$ is a semiflow on Θ, for $\Phi:\Theta\times\mathrm{R}_+\to B(X)$, $\Phi(\theta,t)x=e^{\int_0^{t0}\theta(\tau)d\tau}x$ we have that $\pi=(\Phi,\sigma)$ is a linear skew-product semiflow on $E=X\times\Theta$.

Definition 1.3 A linear skew-product semiflow $\pi=(\Phi,\sigma)$ on $E=X\times\Theta$ is said to be **stable** if there exists $N>0$ such that

$$\left\|\Phi(\theta,t)\right\|\leq N\ ,\ \forall(\theta,t)\in\Theta\times\mathrm{R}_+$$

Definition 1.4 A linear skew-product semiflow $\pi=(\Phi,\sigma)$ on $E=X\times\Theta$ is **uniformly exponentially stable** if there are $N,\nu>0$ such that

$$\left\|\Phi(\theta,t)\right\|\leq Ne^{-\nu t}\ ,\ \forall(\theta,t)\in\Theta\times\mathrm{R}_+$$

Example 1.2 Let $\lambda\in\mathrm{R}_+$. Consider the linear skew product semiflow $\pi_\lambda=(\Phi_\lambda,\sigma)$ where $\Phi_\lambda(\theta,t)=e^{-\lambda t}\Phi(\theta,t)$ and $\pi=(\Phi,\sigma)$ is the linear skew product semiflow given in example 1.1 . For $\lambda>\alpha$, π_λ is uniformly exponentially stable and for $\lambda\in[0,\alpha]$ and $\theta_0(\tau)=\alpha$, for all $\tau\geq 0$ we have :

$$\left\|\Phi_{\lambda}\left(\theta_0,t\right)x\right\|=\begin{cases}\|x\|, & if\ \lambda=\alpha\\ e^{\alpha-\lambda}\|x\| & if\ \lambda<\alpha\end{cases}$$, so π_{λ} is not uniformly exponentially stable.

Let $C_b(\mathrm{R}_+,\mathrm{X})$ be the linear space of all bounded continous functions $u:\mathrm{R}_+\to\mathrm{X}$ and

$$C_0\left(\mathrm{R}_+,X\right)=\left\{u\in C_b\left(\mathrm{R}_+,X\right):\lim_{t\to\infty}u\left(t\right)=0\right\}$$

$$C_{00}\left(\mathrm{R}_+,X\right)=\left\{u\in C_b\left(\mathrm{R}_+,X\right):u\left(0\right)=\lim_{t\to\infty}u\left(t\right)=0\right\}$$

Endowed with sup - norm $|\!|\!| u |\!|\!| = \sup_{t\geq 0}\| u(t)\|$, $C_0\left(\mathrm{R}_+,X\right)$ and $C_b\left(\mathrm{R}_+,X\right)$ are Banach spaces.

Let $C_c\left(\mathrm{R}_+,\mathrm{X}\right)$ be the space of all continous functions $u:\mathrm{R}_+\to\mathrm{X}$ with compact support contained in $(0,\infty)$.

Denote by F the linear space of all Bochner measurable functions $u:\mathrm{R}_+\to\mathrm{X}$ identifying the functions which are equal almost everywhere. For every $p\in[1,\infty)$ the linear space

$$L^p\left(\mathrm{R}_+,\mathrm{X}\right)=\left\{u\in F:\int_0^\infty\left\|u\left(t\right)\right\|^p dt<\infty\right\}$$

is a Banach space with respect to the norm:

$$\|u\|_p:=\left(\int_0^\infty\left\|u\left(t\right)\right\|^p dt\right)^{\frac{1}{p}}$$

2. Perron conditions for exponential stability

Definition 2.1 We say that a strongly continous linear skew-product semiflow $\pi=(\Phi,\sigma)$ is $\left(C_c\left(R_+,X\right),B\left(\Theta,C_b\left(R_+,X\right)\right)\right)$ - **stable** if:

(i) the linear operator $P:C_c\left(R_+,X\right)\to B\left(\Theta,C_b\left(R_+,X\right)\right)$

$$P\left(u\right)\left(\theta\right)\left(t\right)=\int_0^t\Phi\left(\tau\left(\theta,\tau\right),t-\tau\right)u\left(\tau\right)d\tau$$

is well defined .

(ii) there exist K>0 such that :

$$\left\|Pu\right\|_{B\left(\Theta,C_b\left(R_+,X\right)\right)}\leq K\,|\!|\!|u|\!|\!|,\forall u\in C_c\left(R_+,X\right)$$

Definition 2.2 Let $E(R_+,X)\in\{C_{00}(R_+,X),C_b(R_+,X)\}$, we say that a strongly continous linear skew-product semiflow $\pi=(\Phi,\sigma)$ is $\left(E(R_+,X),B(\Theta,C_b(R_+,X))\right)$-**stable** if $S:E(R_+,X)\to B(\Theta,C_b(R_+,X))$ with:

$$S(u)(\theta)(t)=\int_0^t\Phi(\sigma(\theta,\tau),t-\tau)u(\tau)\,d\tau$$

is well defined .

Proposition 2.1 If π is $\left(E(R_+,X),B(\Theta,C_b(R_+,X))\right)$-**stable** , then S is a bounded operator.

Proof

Let $(u_n)\subset E(R_+,X)$, $u\in E(R_+,X)$ and $f\in B(\Theta,C_b(R_+,X))$ such that:

$$u_n\underset{n\to\infty}{\longrightarrow}u \text{ in } E(R_+,X) \text{ and } S(u_n)\underset{n\to\infty}{\to}f \text{ in } B(\Theta,C_b(R_+,X))$$

We have that:

$$S(u_n)(\theta)(t)\underset{n\to\infty}{\to}f(\theta)(t),\qquad (\forall)(\theta,t)\in\Theta\times R_+$$

Using $u_n\underset{n\to\infty}{\longrightarrow}u$ uniformly on R_+ we obtain

$$S(u_n)(\theta)(t)=\int_0^t\Phi(\sigma(\theta,\tau),t-\tau)u_n(\tau)\,d\tau\underset{n\to\infty}{\to}\int_0^t\Phi(\sigma(\theta,\tau),t-\tau)u(\tau)\,d\tau=$$
$$=S(u)(\theta)(t),(\forall)(\theta,t)\in\Theta\times R_+$$

We have $f=S(u)$, so S is a closed linear operator and S is bounded.

Proposition 2.2 Let $\pi=(\Phi,\sigma)$ be a linear skew-product semiflow on $E=X\times\Theta$. If there are $t_0>0$ and $c\in(0,1)$ such that $\|\Phi(\theta,t_0)\|\le c$ for $\forall\theta\in\Theta$, than π is uniformly exponentially unstable.

<u>Proof</u>

Let $M\ge1$ and $\omega>0$ be given by definition 1.2. and ν be a pozitive number such that $c=e^{-\nu t_0}$.

Let $\theta\in\Theta$ be fixed. For $t\ge0$ there are $n\in N$ and $r\in[0,t_0]$ such that $t=nt_0+r$. Than we obtain :

$$\|\Phi(\theta,t)\|\le\|\Phi(\sigma(\theta,nt_0),r)\|\|\Phi(\theta,nt_0)\|\le Me^{\omega t_0}e^{-n\nu t_0}\le Ne^{-\nu t}$$

Where $N=Me^{(\omega+\nu)t_0}$. So, π is uniformly exponentially stable.

Theorem 2.1 Let $\pi=(\Phi,\sigma)$ be a strongly continous linear skew-product semiflow on $E=X\times\Theta$. Then the following assertions are equivalent:

(i) π is uniformly exponentially stable;
(ii) π is $\left(C_b(R_+,X),B\left(\Theta,C_b(R_+,X)\right)\right)$- stable;
(iii) π is $\left(C_{00}(R_+,X),B\left(\Theta,C_b(R_+,X)\right)\right)$- stable;
(iv) π is $\left(C_{00}(R_+,X),B\left(\Theta,C_{00}(R_+,X)\right)\right)$- stable;

Proof (i) $\Rightarrow (ii)$ Let N,v > 0 with:

$$\left\|\Phi(\theta,t)\right\|\leq Ne^{-vt} \qquad \forall(\theta,t)\in\Theta\times R_+ .$$

Let $u\in C_b(R_+,X)$. For all $(\theta,t)\in\Theta\times R_+$ we have

$$\left\|S(u)(\theta)(t)\right\|\leq\int_0^t Ne^{-v(t-\tau)}\left\|u(\tau)\right\|d\tau\leq N\left\|u\right\|\int_0^t e^{-vs}ds\leq\frac{N}{v}\left\|u\right\|$$

So $S(u)\in B(\Theta, C_b(\mathbf{R}_+, X))$.

It follows that π is $\left(C_b(R_+,X),B\left(\Theta,C_b(R_+,X)\right)\right)$-stable.

The implication (ii) $\Rightarrow (iii)$ is obvious, $(i)\Rightarrow(iv)$ is a simple exercise, $(iv)\Rightarrow(iii)$ is obvious .

Suppose that (iii) holds and there is $K>0$ such that

$$\left\|P_\theta u\right\|\leq K\left\|u\right\| \tag{2.1}$$

for all $(u,\theta)\in C_o(R_+,X)\times\Theta$.

Consider $M\geq 1$ and $\omega>0$ given by definition 1.2.

Let $\theta\in\Theta$ and $x\in X$. If $\alpha:R_+\to[0,2]$ is a continous functions with the support contained in $(0,1)$ and with the property :

$$\int_0^1\alpha(s)ds=1$$

Then we consider the function

$$u:R_+\to X,\quad u(t)=\alpha(t)\Phi(\theta,t)x$$

So $u\in C_0(R_+,X)$ and $\left\|u\right\|=\sup_{t\in[0,1]}\left\|u(t)\right\|\leq 2Me^{\omega}\left\|x\right\|$

For $t \geq 1$ we have that:

$$P(u)(\theta)(t)=\int_0^t \alpha(s)\Phi(\sigma(\theta,s),t-s)\Phi(\theta,s)x ds=\Phi(\theta,t)x$$

Then using, (2.1)we have:

$$\|\Phi(\theta,t)x\| \leq \|\|P(u)\|\| \leq 2KMe^{\omega}\|x\| \tag{2.2}$$

For $t \in [0,1]$ we have :

$$\|\Phi(\theta,t)\| \leq Me^{\omega} \tag{2.3}$$

So, denoting by $L=(2K+1)Me^{\omega}$ and using (2.2) and (2.3) we obtain:

$$\|\Phi(\theta,t)\| \leq L$$

for all $(\theta,t) \in \Theta \times R_+$

Consider $\nu=\frac{e}{4LK}$ and $\varphi : R_+ \to R_+, \varphi(t)=\int_0^t se^{-\nu s} ds$

The function φ is strictly increasing on R_+ with $\lim_{t\to\infty}\varphi(t)=\frac{1}{\nu^2}$

So, we can choose $\delta > 0$ such that $\varphi(\delta) > \frac{1}{2\nu^2}$.

Let $\theta \in \Theta$ and $x \in X$. Define the function

$$v : R_+ \to X, v(t)=te^{-\nu t}\Phi(\theta,t)x$$

Then $v \in C_{00}(R_+,X)$ and

$$\|\|v\|\| \leq L\|x\| \sup_{t\geq 0} te^{-\nu t} = \frac{L}{\nu e}\|x\|$$

We observe that $P(v)(\theta)(\delta)=\varphi(\delta)\Phi(\theta,\delta)x$ and it follows that :

$$\|\Phi(\theta,\delta)x\| \leq 2\nu^2\varphi(\delta)\|\Phi(\theta,\delta)x\| \leq 2\nu^2\|\|P(v)\|\| \leq 2\nu\frac{LK}{e}\|x\| = \frac{1}{2}\|x\|$$

It results that: $\|\Phi(\theta,\delta)\| \leq \frac{1}{2}$ for all $\theta \in \Theta$. From proposition 2.2 we have that π is uniformly exponentially stable.

Definition 2.3 Let $p \in (1,\infty)$. We say that a strongly continous linear skew-product semiflow $\pi=(\Phi,\sigma)$ is $(C_c(R_+,X), B(\Theta, C_b(R_+,X)))$- **p-stable** if:

(i) the linear operator $P : C_c(R_+,X) \to B(\Theta, C_b(R_+,X))$

$$P(u)(\theta)(t)=\int_0^t \Phi(\tau(\theta,\tau),t-\tau)u(\tau)d\tau$$

is well defined .

(ii) there exist K>0 such that :

$$\|Pu\|_{B(\Theta,C_b(R_+,X))} \le K\|u\|_p, \forall u \in C_c(R_+,X)$$

Definition 2.4 Let $p \in (1,\infty)$, we say that a strongly continous linear skew-product semiflow $\pi = (\Phi,\sigma)$ is $\left(L^p(R_+,X), B(\Theta, C_b(R_+,X))\right)$-**stable** if $Q: L^p(R_+,X) \to B(\Theta, C_b(R_+,X))$ with:

$$Q(u)(\theta)(t) = \int_0^t \Phi(\sigma(\theta,\tau), t-\tau) u(\tau) d\tau$$

is well defined .

Proposition 2.3 If π is $\left(L^p(R_+,X), B(\Theta, C_b(R_+,X))\right)$-stable , then Q is a bounded operator.

Theorem 2.2 Let $p \in (1,\infty)$ and $\pi = (\Phi,\sigma)$ be a strongly continous linear skew-product semiflow on $E = X \times \Theta$. Then the following assertions are equivalent:

(i) π is uniformly exponentially stable;

(ii) π is $\left(L^p(R_+,X), B(\Theta, C_b(R_+,X))\right)$- stable;

(iii) π is $\left(C_c(R_+,X), B(\Theta, C_b(R_+,X))\right)$-p- stable;

Proof

(i) $\Rightarrow$ (ii) Let N,v > 0 such that:

$$\|\Phi(\theta,t)\| \le Ne^{-vt} \qquad \forall (\theta,t) \in \Theta \times R_+ .$$

Let $u \in L^p(R_+,X)$ and $p' = \dfrac{p}{p-1}$. For all $(\theta,t) \in \Theta \times R_+$ we have:

$$\|Q(u)(\theta)(t)\| \le N \int_0^t e^{-v(t-\tau)} \|u(\tau)\| d\tau \le$$

$$\le N \left(\int_0^t e^{-vp'(t-\tau)} d\tau \right)^{1/p'} \left(\int_0^t \|u(\tau)\|^p d\tau \right)^{1/p} \le \frac{N}{(vp')^{1/p'}} \|u\|_p$$

So $Q(u) \in B(\Theta, C_b(R_+,X))$

(ii) $\Rightarrow$ (iii) follows from proposition 2.3.

(iii) $\Rightarrow$ (i) see [7], theorem 2.4.4.

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