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On Asymptotic Behaviors for Linear Skew-Product Semiflows in Banach Spaces

In this paper we give several characterizations of some asymptotic behaviours: stability, instability, dichotomy and trichotomy of linear skew product semiflows in Banach spaces. The obtained results are generalizations of some well-known results on asymptotic behaviours of linear differential equations. There are also presented several examples of semiflows and linear skew-product semiflows in Banach spaces.

1. Linear skew-product semiflow

Let X be a Banach space, let (Θ, d) be a metric space and let $E = X \times \Theta$. We shall denote by $B(X)$ the Banach algebra of all bounded linear operators from X into itself. Throughout the paper, de norm on X and on $B(X)$ will be denoted by $\|\cdot\|$.

Definition 1.1 A continuous mapping $\sigma : \Theta \times \mathrm{R}_+ \to \Theta$ is said to be a **semiflow** on Θ, if it has the following properties:

(i) $\sigma(\theta, 0) = \theta$, $\forall \theta \in \Theta$

(ii) $\sigma(\theta, s + t) = \sigma(\sigma(\theta, s), t)$ for $\forall (\theta, s, t) \in \Theta \times \mathrm{R}_+^2$

Definition 1.2 A pair $\pi = (\Phi, \sigma)$ is called **linear skew-product semiflow** on $E = X \times \Theta$ if σ is a semiflow on Θ and $\Phi : \Theta \times \mathrm{R}_+ \to B(X)$ satisfies the following conditions:

(i) $\Phi(\theta, 0) = I$, the identity operator on X, $\forall \theta \in \Theta$

(ii) $\Phi(\theta, t + s) = \Phi(\sigma(\theta, s), t)\Phi(\theta, s)$ for all $(\theta, t, s) \in \Theta \times \mathrm{R}_+^2$ (the cocycle identity)

(iii) $\exists M \geq 1$ and $\omega > 0$ such that $\|\Phi(\theta, t)\| \leq Me^{\omega t}$ for $\forall (\theta, t) \in \Theta \times \mathrm{R}_+$

If, in addition,

(iv) for every $(x,\theta)\in E$ the mapping $t\to\Phi(\theta,t)x$ is continuous then π is called a strongly continuous linear skew-product semiflow.

Remark 1.1 Statement (iii) is equivalent with the following
(iii)′ there exist a nondecreasing function $f:\mathrm{R}_+\to\mathrm{R}_+^*$ such that $\|\Phi(\theta,t)\|\le f(t)$ for all $(\theta,t)\in\theta\times\mathrm{R}_+$.

Remark 1.2 If $\pi=(\Phi,\sigma)$ is a linear skew-product semiflow on $E=X\times\Theta$ then for every $\lambda\in\mathrm{R}$ the pair $\pi_\lambda=(\Phi_\lambda,\sigma)$, where $\Phi_\lambda(\theta,t)=e^{-\lambda t}\Phi(\theta,t)$ for all $(\theta,t)\in\Theta\times\mathrm{R}_+$, is also a linear skew-product semiflow called *the shifted skew product semiflow* on $E=X\times\Theta$.

Example 1.1
Let X be a Banach space. We consider $C(\mathrm{R}_+,\mathrm{R})$ the space of all continuous functions $f:\mathrm{R}_+\to\mathrm{R}$. This space is metrizable with the metric

$$d(f,g)=\sum_{n=1}^{\infty}\frac{1}{2^n}\cdot\frac{d_n(f,g)}{1+d_n(f,g)} \quad \text{where } d_n(f,g)=\sup_{t\in[0,n]}\left|f(t)-g(t)\right|.$$

Let $a:\mathrm{R}_+\to\mathrm{R}_+$ be a uniformly continuous, decreasing function such that there exists $\alpha:=\lim_{t\to\infty}a(t)>0$. If we denote by $a_s(t)=a(t+s)$ and by $\Theta=\overline{\{a_s:s\in\mathrm{R}_+\}}$ then $\sigma(\theta,t)(s):=\theta(t+s)$ is a semiflow on Θ, for $\Phi:\Theta\times\mathrm{R}_+\to B(X)$, $\Phi(\theta,t)x=e^{\int_0^{t0}\theta(\tau)d\tau}x$ we have that $\pi=(\Phi,\sigma)$ is a linear skew-product semiflow on $E=X\times\Theta$.

2. Stability for Skew – Product Semiflows

Definition 2.1 A linear skew-product semiflow $\pi=(\Phi,\sigma)$ on $E=X\times\Theta$ is said to be **stable** if there exists $N>0$ such that

$$\|\Phi(\theta,t)\|\le N,\ \forall(\theta,t)\in\Theta\times\mathrm{R}_+$$

Definition 2.2 A linear skew-product semiflow $\pi=(\Phi,\sigma)$ on $E=X\times\Theta$ is **uniformly exponentially stable** if there are $N,\nu>0$ such that

$$\|\Phi(\theta,t)\|\le Ne^{-\nu t},\ \forall(\theta,t)\in\Theta\times\mathrm{R}_+$$

Proposition 2.1 Let $\pi = (\Phi, \sigma)$ be a linear skew-product semiflow on $E = X \times \Theta$. If there are $t_0 > 0$ and $c \in (0,1)$ such that $\|\Phi(\theta, t_0)\| \le c$ for $\forall \theta \in \Theta$, than π is uniformly exponentially stable.

Proof

Let $M \ge 1$ and $\omega > 0$ be given by definition 2.2. and ν be a pozitive number such that $c = e^{-\nu t_0}$.

Let $\theta \in \Theta$ be fixed. For $t \ge 0$ there are $n \in \mathrm{N}$ and $r \in [0, t_0]$ such that $t = nt_0 + r$. Than we obtain :

$$\|\Phi(\theta, t)\| \le \|\Phi(\sigma(\theta, nt_0), r)\| \|\Phi(\theta, nt_0)\| \le Me^{\omega t_0} e^{-n\nu t_0} \le Ne^{-\nu t}$$

Where $N = Me^{(\omega+\nu)t_0}$. So, π is uniformly exponentially stable.

Definition 2.3 A linear skew-product semiflow $\pi = (\Phi, \sigma)$ on $E = X \times \Theta$ is said to be **unstable** if there exists $N > 0$ such that

$\|\Phi(\theta, t)\| \ge N\|x\|, \ \forall (x, \theta, t) \in E \times \mathrm{R}_+$

Definition 2.4 A linear skew-product semiflow $\pi = (\Phi, \sigma)$ on $E = X \times \Theta$ is said to be **uniformly exponentially unstable** if there exists $N > 0$ such that

$\|\Phi(\theta, t)\| \ge Ne^{\nu t}\|x\|, \ \forall (x, \theta, t) \in E \times \mathrm{R}_+$

Proposition 2.2 Let $\pi = (\Phi, \sigma)$ be a linear skew-product semiflow on $E = X \times \Theta$. If there are $t_0 > 0$ and $\delta > 1$ such that $\|\Phi(\theta, t_0)x\| \ge \delta\|x\|, \ \forall (x, \theta) \in E$ then π is uniformly exponentially unstable.

Proof:

Let $M \ge 1, \omega > 0$ be given by definition 2.2. and $\nu > 0$ such that $\delta = e^{\nu t_0}$. Let $(x, \theta) \in E$. For $t \ge 0$ there is $k \in \mathrm{N}$ and $r \in [0, t_0)$ such that $t = kt_0 + r$. Using the cocycle identity and the hypothesis, it follows that

$$\delta^{k+1}\|x\| \le \|\Phi(\theta, (k+1)t_0)x\| \le Me^{\omega t_0}\|\Phi(\theta, t)x\|.$$

Denoting $N = \dfrac{1}{Me^{\omega t_0}}$, we deduce that

$\|\Phi(\theta, t)x\| \ge Ne^{\nu t}\|x\|, \ \forall (x, \theta, t) \in E \times \mathrm{R}_+$

so π is uniformly exponentially unstable.

3. Exponential Dichotomy for Linear Skew-Product Semiflow

Definition 3.1 A mapping **P**$: E \to E$ is said to be a **projector** if **P** is continuous and has the form **P**$(\theta,t) = (P(\theta)x,\theta)$ (3.1)

where $P(\theta)$ is a bounded linear projection on X.

Remark 3.1 $P(\theta): E \to E$ is a bounded linear mapping with the property $P(\theta)P(\theta) = P^2(\theta) = P(\theta)$ for all $\theta \in \Theta$.

Definition 3.2 A projector **P** on E is said to be **invariant** if it satisfies the following property:

$$P(\sigma(\theta,t))\Phi(\theta,t) = \Phi(\theta,t)P(\theta),\ t \geq 0, \theta \in \Theta \quad (3.2)$$

Definition 3.3 The mapping **Q**$: E \to E$ given by

$$\mathbf{Q}(x,\theta) = (x - P(\theta)x,\theta) \quad (3.3)$$

where P is a linear projection on X, is called the **complementary projector** to **P** on E.

For any subset $F \subset E$ we have $F(\theta) := \{x \in X : (x,\theta) \in F\},\ \theta \in \Theta$.So $E(\theta) = X,\ \theta \in \Theta$.

Lema 3.1 Let **P** be a projector on E. Then R and N are closed subsets in E and we have: $R(\theta) \cap N(\theta) = \{0\}, R(\theta) + N(\theta) = E(\theta)$ for all $\theta \in \Theta$.

Remark 3.2 One has $R(Q) = N(P)$ and $N(Q) = R(P)$.

Definition 3.4 We say that a linear skew-product semiflow $\pi = (\Phi,\sigma)$ on E has an **exponential dichotomy** over Θ, if there are constants $K \geq 1, \nu > 0$ and invariant projector **P** such that for all $\theta \in \Theta$ we have the following:

(1) $\Phi(\theta,t): N(P(\theta)) \to N(P(\sigma(\theta,t)))$ is an isomorfism, with inverse

$\Phi(\sigma(\theta,t),-t): N(P(\sigma(\theta,t))) \to N(P(\theta)),\ \ t \geq 0$

(2) $\|\Phi(\theta,t)P(\theta)\| \leq Ke^{-\nu t}$, $t \geq 0$ (3.4)

(3) $\|\Phi(\theta,t)(I - P(\theta))\| \geq Ke^{\nu t}$, $t \leq 0$ (3.5)

Definition 3.4 A linear skew-product semiflow $\pi = (\Phi,\sigma)$ is said to be **uniformly exponentially dichotomic** if there exist a family of projections $\{P(\theta)\}_{\theta\in\Theta} \subset B(X)$ and two constants $K \geq 1$ and $\nu > 0$ such that

(i) $\Phi(\theta,t)P(\theta) = P(\sigma(\theta,t))\Phi(\theta,t)$, for all $(\theta,t) \in \Theta \times \mathrm{R}_+$

(ii) $\left\|\Phi(\theta,t)x\right\| \le Ke^{-\nu t}\left\|x\right\|$, for all $x \in R\left(P(\theta)\right)$ and all $(\theta,t) \in \Theta \times \mathrm{R}_+$

(iii) $\left\|\Phi(\theta,t)x\right\| \ge \dfrac{1}{K}e^{\nu t}\left\|x\right\|$, for all $x \in N\left(P(\theta)\right)$ and all $(\theta,t) \in \Theta \times \mathrm{R}_+$

(iv) the restriction $\Phi(\theta,t)_| : N\left(P(\theta)\right) \to N\left(P\left(\sigma(\theta,t)\right)\right)$ is an isomorfism, for every $(\theta,t) \in \Theta \times \mathrm{R}_+$

Proposition 3.1 Let $\pi = (\Phi, \sigma)$ be a strongly continuous skew-product on $E = X \times \Theta$. If π is uniformly exponentially dichotomic relative to the family of projections $\{P(\theta)\}_{\theta\in\Theta}$ then

(i) $\sup\limits_{\theta\in\Theta}\| P(\theta)\| < \infty$

(ii) for every $(\theta,t) \in \Theta \times \mathrm{R}_+^*$ and every $x \in N\left(\mathrm{P}(\sigma(\theta,t))\right)$ the mapping $s \to \Phi\left(\sigma(\theta,s), t-s\right)_|^{-1} x$ is continuous on $[\theta, t]$

(iii) for every $(x,\theta) \in E$ the mapping $t \to P\left(\sigma(\theta,t)\right)x$ is continuous on R_+.

Proof

(i) For every $\theta \in \Theta$ we define

$\delta_\theta := \inf\left\{\|x_1 + x_2\| : x_1 \in R\left(P(\theta)\right),\ x_2 \in N\left(P(\theta)\right),\ \|x_1\| = \|x_2\| = 1\right\}$.

Let $\theta \in \Theta$ and $x \in X$ with $P(\theta)x \ne 0$ and $\left(I - P(\theta)\right)x \ne 0$.

Then $$\delta_\theta \le \left\| \frac{P(\theta)x}{\|P(\theta)x\|} + \frac{\left(I-P(\theta)\right)x}{\left\|\left(I-P(\theta)\right)x\right\|} \right\| =$$

$$\frac{1}{\|P(\theta)x\|}\left\| x + \frac{\|P(\theta)x\| - \left\|\left(I-P(\theta)\right)x\right\|}{\left\|\left(I-P(\theta)\right)x\right\|}\left(I-P(\theta)\right)x \right\| \le \frac{2\|x\|}{\|P(\theta)x\|}$$

It results that $\|\mathrm{P}(\theta)x\| \le \dfrac{2}{\delta_\theta}$, for all $\theta \in \Theta$.

If $x_1 \in R\left(P(\theta)\right)$ and $x_2 \in N\left(P(\theta)\right)$ such that $\|x_1\| = \|x_2\| = 1$, then, for every $t \ge 0$ we have

$$\|x_1 + x_2\| \ge \frac{1}{\mathrm{M}}e^{-\omega t}\|\Phi(\theta,t)x_1 + \Phi(\theta,t)x_2\| \ge \frac{1}{\mathrm{M}}e^{-\omega t}\left(\frac{1}{K}e^{\nu t} - Ke^{-\nu t}\right)$$

where M, ω are given by Definition 2.2 and K, ν are given by Definition 3.4. It follows that there is $c > 0$ such that $\delta_\theta \ge c$, for all $\theta \in \Theta$.

(ii) Let $t > 0$, $\theta \in \Theta$ and $x \in N\left(P\left(\sigma\left(\theta,t\right)\right)\right)$. There is $y \in N\left(P\left(\theta\right)\right)$ such that $x = \Phi\left(\theta,t\right)y$.

Let $s_0 \in \left[0,t\right]$. It is easy to see that

$$\Phi(\sigma(\theta,s),t-s)_{|}^{-1}x - \Phi(\sigma(\theta,s_0),t-s_0)_{|}^{-1}x = \Phi(\theta,s)y - \Phi(\theta,s_0)y \xrightarrow[s\to s_0]{} 0$$

(iii) Let $\left(x,\theta\right) \in E$. Let $t_0 > 0$. We have that

$$\begin{aligned}\left\|P\left(\sigma,\left(\theta,t\right)\right)x - P\left(\sigma\left(\theta,t_0\right)\right)x\right\| &\leq \left\|P\left(\sigma\left(\theta,t\right)\right)x - P\left(\sigma\left(\theta,t\right)\right)\Phi\left(\sigma\left(\theta,t_0\right),t-t_0\right)x\right\| + \\ &\left\|\Phi\left(\sigma\left(\theta,t_0\right),t-t_0\right)P\left(\sigma\left(\theta,t_0\right)\right)x - P\left(\sigma\left(\theta,t_0\right)\right)x\right\| \leq \\ &\sup_{\theta\in\Theta}\left\|P\left(\theta\right)\right\| \cdot \left\|\Phi\left(\sigma\left(\theta,t_0\right),t-t_0\right)x - x\right\| + \\ &\left\|\Phi\left(\sigma\left(\theta,t_0\right),t-t_0\right)P\left(\sigma\left(\theta,t_0\right)\right)x - P\left(\sigma\left(\theta,t_0\right)\right)x\right\| \to 0\end{aligned}$$

as $t \,\square\, t_0$, so the mapping $P\left(\sigma\left(\theta,\square\right)\right)x$ is right-continuous in t_0 .

Let $t < t_0$. Since

$$\left(I - P\left(\sigma\left(\theta,t\right)\right)\right)x = \Phi\left(\sigma\left(\theta,t\right),t_0-t\right)^{-1}\Phi\left(\sigma\left(\theta,t\right),t_0-t\right)\left(I - P\left(\sigma\left(\theta,t\right)\right)\right)x =$$

$$\Phi\left(\sigma\left(\theta,t\right),t_0-t\right)^{-1}\left(I - P\left(\sigma\left(\theta,t_0\right)\right)\right)\Phi\left(\sigma\left(\theta,t\right),t_0-t\right)x \to \left(I - P\left(\sigma\left(\theta,t_0\right)\right)\right)x$$

as $t \,\square\, t_0$, we obtain that the mapping $P\left(\sigma\left(\theta,\square\right)\right)x$ is left-continuous in t_0 .

Example 3.1

Let us consider $X = \mathrm{R}^2$ with the norm $\left\|\left(x_1,x_2\right)\right\| = \left|x_1\right| + \left|x_2\right|$ .We denote $C\left(\mathrm{R}_+,\mathrm{R}_+\right)$ the set of all continuous functions $f:\mathrm{R}_+ \to \mathrm{R}_+$. This space is metrizable with the metric: $d\left(f,g\right) = \sum_{n=1}^{\infty}\frac{1}{2^n}\cdot\frac{d_n\left(f,g\right)}{1+d_n\left(f,g\right)}$ where $d_n\left(f,g\right) = \sup_{t\in[0,n]}\left|f\left(t\right) - g\left(t\right)\right|$.

If $f \in C\left(R_+,R\right)$ then for every $t \in \mathrm{R}_+$ we denote by $f_s\left(t\right) = f\left(t+s\right)$. Let us consider $\Theta = \left\{a_s, s \in \mathrm{R}_+\right\}$ where $a:\mathrm{R}_+ \to \mathrm{R}_+^*$ is a decreasing function with $\alpha := \lim_{t\to\infty} a\left(t\right) > 0$. Then $\left(\Theta,d\right)$ is a metric space and $\sigma\left(\theta,t\right)(s) = f\left(t+s\right)$ is a semiflow on Θ . Then $\Phi:\Theta\times\mathrm{R}_+ \to B\left(X\right)$ is given by

$$\Phi\left(\theta,t\right)\left(x_1,x_2\right) = \left(e^{-2ta(0)+\int_0^t f(s)ds}x_1,\, e^{\int_0^t f(s)ds}x_2\right)$$

and we have that $\pi = (\Phi,\sigma)$ is a linear skew-product semiflow on $E = X\times\Theta$. We consider the projections:

$P\left(x\right)\left(x_1,x_2\right) = \left(x_1,0\right)$, $\qquad Q\left(x\right)\left(x_1,x_2\right) = \left(0,x_2\right)$

Following relations hold:

$$\|\Phi(\theta,t)P(\theta)x\| \le e^{-t\alpha(0)}\|P(\theta)x\|, \qquad \|\Phi(\theta,t)Q(\theta)x\| \ge e^{\alpha t}\|Q(\theta)x\|$$

which proves that the linear skew-product semiflow $\pi=(\Phi,\sigma)$ is uniformly exponentially dichotomic.

4. Exponential Trichotomy Of Linear Skew-Product Semiflows in Banach Spaces

Definition 4.1 We say that a linear skew-product semiflow $\pi=(\Phi,\sigma)$ has **uniform exponential trichotomy** on *E* if there exist three families of projections $(P_0(\theta))_{\theta\in\Theta}$, $(P_1(\theta))_{\theta\in\Theta}$, $(P_2(\theta))_{\theta\in\Theta}$ with characteristics $N_0, N_1, N_2 \ge 1$, $\nu_1,\nu_2 > 0$ such that:

(i) $P_i(\theta)P_j(\theta)=0$ for $\forall\ i \ne j$, $i, j \in \{0,1,2\}$ and $\forall\ \theta\in\Theta$

$P_0(\theta)+P_1(\theta)+P_2(\theta)=I$ for $\forall\ \theta\in\Theta$

(ii) $\Phi(\theta,t)P_j(\theta)=P_j(\sigma(\theta,t))\Phi(\theta,t)$ for $\forall\ (\theta,t)\in\Theta\times\mathrm{R}_+$ and $\forall\ x\in X$

(iii) $\|P_0(\theta)x\| \le N_0\|\Phi(\theta,t)P_0(\theta)x\| \le N_0^2\|P_0(\theta)x\|$ for $\forall\ (\theta,t)\in\Theta\times\mathrm{R}_+$ and $\forall\ x\in X$

(iv) $\|\Phi(\theta,t)P_1(\theta)x\| \le N_1 e^{-\nu_1 t}\|P_1(\theta)x\|$ for $\forall\ (\theta,t)\in\Theta\times\mathrm{R}_+$ and $\forall\ x\in X$

(v) $N_2\|\Phi(\theta,t)P_2(\theta)x\| \ge e^{\nu_2 t}\|P_2(\theta)x\|$ for $\forall\ (\theta,t)\in\Theta\times\mathrm{R}_+$ and $\forall\ x\in X$

Remark 4.1 If we denote $N=\max\{N_0,N_1,N_2\}$ and $\nu=\min\{\nu_1,\nu_2\}$ we have that in definition 1 we can assume that $N_0=N_1=N_2=N$ and $\nu_1=\nu_2=\nu$.

Example 4.1 Let $X=\mathrm{R}^3$ with the norm $\|(z_1,z_2,z_3)\|=|z_1|+|z_2|+|z_3|$

Let $C=C(\mathrm{R}_+,\mathrm{R}_+)$ continue the set of all continuous functions $x:\mathrm{R}_+\to\mathrm{R}_+$. This space is metrizable with the metric:

$$d(x,y)=\sum_{n=1}^{\infty}\frac{1}{2^n}\cdot\frac{d_n(x,y)}{1+d_n(x,y)} \qquad \text{unde } d_n(x,y)=\sup_{t\in[0,n]}|x(t)-y(t)|$$

If $x\in C$ then for every $t\in\mathrm{R}_+$ we denote by $x_t\in C$ the function $x_t(s)=x(t+s)$. Let us consider $\Theta=\overline{\{f_t, t\in\mathrm{R}_+\}}$, where $f:\mathrm{R}_+\to\mathrm{R}_+^*$ is a decreasing function with $\lim_{t\to\infty} f(t)=\alpha>0$.

Then (Θ, d) is a metric space and $\sigma : \Theta \times \mathrm{R}_+ \to \Theta$, $\sigma(x,t)(s) = x(t+s)$ is a semiflow on X and $\Phi : \Theta \times \mathrm{R}_+ \to B(X)$ is given by :

$$\Phi(x,t)(z_1, z_2, z_3) = \left(e^{-2tf(0)+\int_0^t x(s)ds} z_1, e^{\int_0^t x(s)ds} z_2, e^{-tf(0)+2\int_0^t x(s)ds} z_3 \right)$$

then $\pi = (\Phi, \sigma)$ is a linear skew-product semiflow on $E = X \times \Theta$. We consider the projections :

$P_1(x)(z_1, z_2, z_3) = (z_1, 0, 0)$

$P_2(x)(z_1, z_2, z_3) = (0, z_2, 0)$, $P_3(x)(z_1, z_2, z_3) = (0, 0, z_3)$

We have $\|\Phi(x,t)P_1(x)z\| \le e^{-tf(0)} \|P_1(x)z\|$ (4.1)

$$\|\Phi(x,t)P_2(x)z\| \le e^{\alpha t} \|P_2(x)z\| \tag{4.2}$$

$$e^{-tf(0)} \|P_3(x)z\| \le \|\Phi(x,t)P_3(x)z\| = e^{tf(0)} \|P_3(x)z\| \tag{4.3}$$

which proves that the linear skew-product semiflow $\pi = (\Phi, \sigma)$ is uniformly exponentially trichotomic.

Proposition 4.1 A linear skew-product semiflow $\pi = (\Phi, \sigma)$ is uniformly exponentially trichotomic if and only if there exist the constants $N_0, N_1, N_2 \ge 1$, $\nu_1, \nu_2 > 0$ and three families of projections $(P_0(\theta))_{\theta \in \Theta}$, $(P_1(\theta))_{\theta \in \Theta}$, $(P_2(\theta))_{\theta \in \Theta}$ such that :

(i)′ $P_i(\theta)P_j(\theta) = 0$; $P_0(\theta) + P_1(\theta) + P_2(\theta) = I$ for $\forall\ i \ne j$ şi $\forall\ \theta \in \Theta$

(ii)′ $\Phi(\theta,t)P_j(\theta) = P_j(\sigma(\theta,t))\Phi(\theta,t)$ for $\forall\ (\theta,t) \in \Theta \times \mathrm{R}_+$ and $\forall\ x \in X$

(iii)′ $\|\Phi(\theta,t_0)P_0(\theta)x\| \le N_0 \|\Phi(\theta,t+t_0)P_0(\theta)x\| \le N_0^2 \|\Phi(\theta,t_0)P_0(\theta)x\|$ for $\forall\ (\theta,t,t_0) \in \Theta \times \mathrm{R}_+^2$ and $\forall\ x \in X$

(iv) $\|\Phi(\theta,t+t_0)P_1(\theta)x\| \le N_1 e^{-\nu_1 t} \|\Phi(\theta,t_0)P_1(\theta)x\|$ for $\forall\ (\theta,t,t_0) \in \Theta \times \mathrm{R}_+^2$ and $\forall\ x \in X$

(v)′ $N_2 \|\Phi(\theta,t+t_0)P_2(\theta)x\| \ge e^{\nu_2 t} \|\Phi(\theta,t_0)P_2(\theta)x\|$ for $\forall\ (\theta,t,t_0) \in \Theta \times \mathrm{R}_+^2$.

Proof

Necessity

The conditions (i)′ and (ii)′ are exactly (i) and (ii) from definition 1.

We will prove first (iii)′

$\|\Phi(\theta,t_0)P_0(\theta)x\| = \|P_0(\sigma(\theta,t_0)\Phi(\theta,t_0)x)\| \le$

$N_0 \|\Phi(\sigma(\theta,t_0),t)P_0(\sigma(\theta,t_0))\Phi(\theta,t_0)x\| =$

$$= N_0 \left\| \Phi(\sigma(\theta,t_0),t)\Phi(\theta,t_0)P_0(\theta)x \right\| = N_0 \left\| \Phi(\theta,t+t_0)P_0(\theta)x \right\|$$

$$= N_0 \left\| \Phi(\sigma(\theta,t_0),t)\Phi(\theta,t_0)P_0(\theta)x \right\| = N_0 \left\| \Phi(\sigma(\theta,t_0),t)P_0(\sigma(\theta,t_0)\Phi(\theta,t_0))x \right\| \leq$$

$$\leq N_0^2 \left\| P_0(\sigma(\theta,t_0))\Phi(\theta,t_0)x \right\| = N_0^2 \left\| \Phi(\theta,t_0)P_0(\theta)x \right\|$$

Using (iv) from definition 1 we have :

$$\left\| \Phi(\theta,t+t_0)P_1(\theta)x \right\| = \left\| \Phi(\sigma(\theta,t_0),t)\Phi(\theta,t_0)P_1(\theta)x \right\|$$

$$= \left\| \Phi(\sigma(\theta,t_0),t)P_1(\sigma(\theta,t_0)\Phi(\theta,t_0)x) \right\|$$

$\leq N_1 e^{-\nu_1 t} \left\| P_1(\sigma(\theta,t_0))\Phi(\theta,t_0)x \right\| = N_1 e^{-\nu_1 t} \left\| \Phi(\theta,t_0)P_1(\theta)x \right\|$ and we obtain (iv').

Simillary

$$N_2 \left\| \Phi(\theta,t+t_0)P_2(\theta)x \right\| = N_2 \left\| \Phi(\sigma(\theta,t_0),t)\Phi(\theta,t_0)P_2(\theta)x \right\|$$

$$= N_2 \left\| \Phi(\sigma(\theta,t_0),t)P_2(\sigma(\theta,t_0))\Phi(\theta,t_0)x \right\| \geq e^{\nu_2 t} \left\| P_2(\sigma(\theta,t_0))\Phi(\theta,t_0)x \right\|$$

$$= e^{\nu_2 t} \left\| \Phi(\theta,t_0)P_2(\theta)x \right\|$$

Sufficiency is trivial.

Proposition 4.2 The conditions (iv)' and (v)' from propositions 1 are equivalent with :

(iv)'' there exists a function $f : \mathrm{R}_+ \to (0,\infty)$ cu $\lim_{t\to\infty} f(t) = 0$ such that:

$$\left\| \Phi(\theta,t+t_0)P_1(\theta)x \right\| \leq f(t) \left\| \Phi(\theta,t_0)P_1(\theta)x \right\|$$

for all $(\theta,t,t_0) \in \Theta \times \mathrm{R}_+^2$ and all $x \in X$

(v)' there exist a function $g : \mathrm{R}_+ \to (0,\infty)$ with $\lim_{t\to\infty} g(t) = \infty$ such that :

$$\left\| \Phi(\theta,t+t_0)P_2(\theta)x \right\| \geq g(t) \left\| \Phi(\theta,t_0)P_2(\theta)x \right\|$$

for all $(\theta,t,t_0) \in \Theta \times \mathrm{R}_+^2$ and all $x \in X$.

Proposition 4.3

A linear skew-product semiflow π is uniformly exponentialyl trichotomic if and only if there exist the constants $N_1, N_2, N_3, N_4 \geq 1$, $\nu_1, \nu_2 > 0$ and two families of projectors $(P(\theta))_{\theta\in\Theta}$, $(Q(\theta))_{\theta\in\Theta}$ such that :

(i) $P(\theta)Q(\theta) = Q(\theta)P(\theta)$, for all $\theta \in \Theta$

(ii) $\Phi(\theta,t)P(\theta) = P(\sigma(\theta,t))\Phi(\theta,t)$

$\Phi(\theta,t)Q(\theta) = Q(\sigma(\theta,t))\Phi(\theta,t)$ for $\forall\ \theta \in \Theta$ and $t \in \mathrm{R}_+$.

(iii) $\left\| \Phi(\theta,t+t_0)P(\theta)x \right\| \leq N_1 e^{-\nu_1 t} \left\| \Phi(\theta,t_0)P(\theta)x \right\|$ for $\forall\ (\theta,t,t_0) \in \Theta \times \mathrm{R}_+^2$ and $\forall\ x \in X$

(iv) $N_2\|\Phi(\theta,t+t_0)Q(\theta)x\|\geq e^{\nu_2 t}\|\Phi(\theta,t_0)Q(\theta)x\|$ for $\forall\ (\theta,t,t_0)\in\Theta\times R_+^2$ and $\forall\ x\in X$

(v) $\|\Phi(\theta,t+t_0)(I-Q(\theta)x\|\leq N_3\|\Phi(\theta,t_0)(I-Q(\theta))x\|$ for $\forall\ (\theta,t,t_0)\in\Theta\times R_+^2$ and $\forall\ x\in X$

(vi) $N_4\|\Phi(\theta,t+t_0)(I-P(\theta))x\|\geq\|\Phi(\theta,t_0)(I-P(\theta))x\|$ for $\forall\ (\theta,t,t_0)\in\Theta\times R_+^2$ and $\forall\ x\in X$

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